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ON AN APPROXIMATION FORMULA

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1. INTRODUCTION

This Note contains some remarks concerning an approximation formula for functions, which is a generalization of some interpolation formulae given in [2] and [4]. In particular, we shall show that only one of the formulae of this type, mentioned in [4], has a maximal degree of exactness. Some particular cases of such formulae were also mentioned in [4, p. 163].

Denote by I_x the closed interval determined by two distinct points x_0, x in $\mathbf{R}$. For a $(2n+1)$-times derivable function $f: I_x \to \mathbf{R}$ and $n \in \mathbf{N}$, consider the class G of functions given by

$$G = \Big\{ g : g(t) = f(x_0) + (t - x_0) \sum_{i=1}^{n} a_i f'\big(x_0 + b_i(t - x_0)\big), \quad a_i, b_i \in \mathbf{R}, i = \overline{1,n}, t \in I_x \Big\}. \tag{1.1}$$

Consider the following problem: Find a function $\bar{g} \in G$ such that

$$f^{(i)}(x_0) = \bar{g}^{(i)}(x_0), \; i = \overline{1,m}. \tag{1.2}$$

In [4] this problem was solved in some particular cases. We shall show that, for $m = 2n$, this problem has a unique solution and we shall give a representation for the remainder.

2. DETERMINATION OF THE APPROXIMATING FUNCTION

For $m = 2n$, we are looking for a function $\bar{g}$ in G verifying conditions (1.2) and having a maximal degree of approximation.

It is easily seen that conditions (1.2) lead to the following system, having the real numbers a_i, b_i, $i = \overline{1,n}$, as unknowns:

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$$\sum_{i=1}^{n} a_i b_i^k = 1/(k+1),\ k = 0,1,\dots,2n-1. \tag{2.1}$$

Consider now a continuous function $\varphi:[0,1]\to\mathbb{R}$ and let

$$\int_0^1 \varphi(t)\mathrm{d}t = \sum_{i=1}^{n} a_i\varphi(b_i) + R[\varphi] \tag{2.2}$$

be a quadrature formula, having $\{b_i\}_1^n$ as knots and $\{a_i\}_1^n$ as coefficients. Asking that $R[\varphi_k] = 0$ for $\varphi_k(t) = t^k$, $k = \overline{0, 2n-1}$, formula (2.2) becomes the classical Gauss quadrature formula.

On the other hand, the conditions $R[\varphi_k] = 0$, for $\varphi_k(t) = t^k$, $k = \overline{0,2n-1}$, lead again to the system (2.1), implying that b_i must be the roots of the Legendre polynomial w_n of degree n, i.e., the roots of the equation

$$w_n(t) := \frac{n!}{(2n)!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[t^n(t-1)^n\right] = 0. \tag{2.3}$$

The coefficients a_i are given by the following formula

$$a_i = \frac{(n!)^4}{[(2n)!]^2 b_i(1-b_i)[w'_n(b_i)]^2},\ i = \overline{1,n}, \tag{2.4}$$

(see [1], p. 261).

Now, it is clear that the following theorem holds:

THEOREM 2.1. *If $f: I_x \to \mathbb{R}$ is a $(2n+1)$-times derivable function on I_x, then there exists only one function $\bar{g} \in G$ verifying conditions* (1.2) *for $m = 2n$. The parameters $\{a_i\}_{i=1}^n$ are given by formula* (2.4), *where $\{b_i\}_{i=1}^n$ are the roots of equation* (2.3).

3. DETERMINATION OF THE REMAINDER

Consider the approximation formula

$$f(x) = \bar{g}(x) + r[f], \tag{3.1}$$

where $\bar{g} \in G$ is a function verifying (2.1) and $r[f]$ is the remainder.

In the conditions of Theorem 2.1, it follows that

$$f'(x_0 + b_i(x-x_0)) = \sum_{j=1}^{2n}\frac{f^{(j)}(x_0)}{(j-1)!}b_i^{j-1}(x-x_0)^{j-1} + r_i(x), \tag{3.2}$$

where

$$r_i(x) = \frac{f^{(2n+1)}(\theta_i)}{(2n)!}b_i^{2n}(x-x_0)^{2n}, \tag{3.3}$$

and θ_i is a number contained in the open interval determined by x_0 and $x_0 + b_i(x-x_0)$, $1 \le i \le n$.

From (3.2) we obtain the equalities

$$f(x) - f(x_0) - (x-x_0)f'(x_0 + b_i(x-x_0)) = \tag{3.4}$$

$$= f(x) - f(x_0) - \sum_{j=1}^{2n}\frac{f^{(j)}(x_0)}{(j-1)!}b_i^{j-1}(x-x_0)^j - r_i(x)(x-x_0),\ i = \overline{1,n}.$$

Multiplying equalities (3.4) by a_i, taking into account solutions (2.1) and summing up, we obtain

$$f(x) - \bar{g}(x) = f(x) - \sum_{j=0}^{2n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j - \sum_{i=1}^{n} a_i r_i(x)(x-x_0). \tag{3.5}$$

Now, using (3.3) and Lagrange form of the remainder in the Taylor formula, we get

$$f(x) - \bar{g}(x) = \left[\frac{f^{(2n+1)}(\eta)}{(2n+1)!} - \sum_{i=1}^{n} a_i b_i^{2n}\frac{f^{(2n+1)}(\theta_i)}{(2n)!}\right](x-x_0)^{2n+1}, \tag{3.6}$$

where $\eta \in I_x$.

Setting $\varphi(t) = t^{2n}$ in (2.2) and taking into account the form of the remainder term in the Gauss quadrature formula [1, p. 259], we get

$$\sum_{i=1}^{n} a_i b_i^{2n} + \frac{[n!]^4}{[(2n)!]^2(2n+1)} = \frac{1}{2n+1},$$

implying

$$\sum_{i=1}^{n} a_i b_i^{2n} = \frac{[(2n)!]^2 - [n!]^4}{(2n+1)[(2n)!]^2} \tag{3.7}$$

Suppose now that the $(2n+1)$-order derivative of f is bounded on I_x and let

$$M_{2n+1} = \sup_{t \in I_x} \left| f^{(2n+1)}(t) \right|. \tag{3.8}$$

Taking into account relations (3.6) and (3.7), one obtains the following delimitation for $r[f]$

$$\left| r[f] \right| \leq \frac{M_{2n+1}}{(2n+1)!} \cdot \frac{2 \cdot [(2n)!]^2 + [n!]^4}{[(2n)!]^2} \left| x - x_0 \right|^{2n+1}. \tag{3.9}$$

4. PARTICULAR CASES

a) $n = 1$. In this case, $b_1 = 1/2$, $a_1 = 1$ and

$$g(x) = f(x_0) + (x - x_0) f'\left(x_0 + \frac{1}{2}(x - x_0) \right).$$

From (3.9) we get

$$\left| f(x) - g(x) \right| \leq \frac{7M_3}{24} \left| x - x_0 \right|^3,$$

where $M_3 = \sup_{t \in I_x} \left| f'''(t) \right|$.

b) $n = 2$. In this case, $b_1 = \frac{3 - \sqrt{3}}{6}$, $b_2 = \frac{3 + \sqrt{3}}{6}$, $a_1 = a_2 = \frac{1}{2}$ and

$$g(x) = f(x_0) + \frac{1}{2}(x - x_0)\left[f'\left(x_0 + \frac{3 - \sqrt{3}}{6}(x - x_0) \right) + f'\left(x_0 + \frac{3 + \sqrt{3}}{6}(x - x_0) \right) \right].$$

One also obtains the evaluation

$$\left| f(x) - g(x) \right| \leq \frac{71M_5}{4320} \left| x - x_0 \right|^5,$$

where $M_5 = \sup_{t \in I_x} \left| f^{(5)}(t) \right|$.

Remark. Approximation formula of the type considered in this Note could be useful for the approximate calculation of the values of some functions having rational functions as derivatives.

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