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A CERTAIN CLASS OF LINEAR OPERATORS

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1. INTRODUCTION

In this paper we define for each $n \in \mathbf{N}$, $n \geq 1$, the operator

$$R_{n,a} = L_n(-aD),$$

where $a \in \mathbf{C} \setminus \{0\}$ and $L_n(t) = L_n^{(0)}(t)$ are simple Laguerre polynomials. For these linear operators with the indicator $f(t) = L_n(-at)$ it is given a theorem of representation using the Pincherle derivative of n-order, it is found the differential operator equation verified by $R_{n,a}$, and it is given the generating function.

Finally, we study the indicator of the delta operator $Q_{n,a} = R_{n,a} - I$, $n \geq 1$, giving a formula for $Q'^{-1}_{n,a}$.

2. PRELIMINARIES AND RESULTS

Let $L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n;1+\alpha;x)$ and $L_n(x) = L_n^{(0)}(x)$, $l_n(x) = (1)_n L_n^{(-1)}(x)$ be the Laguerre polynomials. Let Π be the linear space of polynomials with complex coefficients and let Π^* be the space of linear operators $T:\Pi \to \Pi$. For each $n \in \mathbf{N}^*$ we define the operator

(1) $$R_{n,a} = L_n(-aD),$$

where $a \in \mathbf{C} \setminus \{0\}$ and D is the derivative operator.

THEOREM 1. *$R_{n,a}$ is given by*

(2) $$R_{n,a} = \frac{1}{n!}\left(D^n E^a\right)^{(n)} E^{-a},$$

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where E^a is the shift operator and $\left(D^nE^a\right)^{(n)}$ is the Pincherle derivative of n-order.

Proof. We have

$$R_{n,a}=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k!}(aD)^k=\frac{1}{n!}\sum_{k=0}^{n}\frac{\langle n\rangle_k\langle n\rangle_{n-k}}{k!}a^kD^k,$$

where $\langle i\rangle_k=i(i-1)\dots(i-k+1)$.

We can prove by complete induction that the Pincherle derivative of n-order of the composition TS is given by

$$(TS)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}T^{(n-k)}S^{(k)}$$

and hence

$$\left(D^nE^a\right)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}\langle n\rangle_{n-k}a^kD^kE^a,$$

$$\left(D^nE^a\right)^{(n)}E^{-a}=\sum_{k=0}^{n}\frac{\langle n\rangle_k\langle n\rangle_{n-k}}{k!}a^kD^k,$$

so

$$R_{n,a}=\frac{1}{n!}\left(D^nE^a\right)^{(n)}E^{-a}.$$

THEOREM 2. *We have*

$$\sum_{k=0}^{n}\left(R_{n,a}x^k\right)(0)=(1+a)^n.$$

Proof. We consider the following

$$R_{n,a}=\frac{1}{n!}\sum_{k=0}^{n}\frac{\langle n\rangle_k\langle n\rangle_{n-k}a^k}{k!}D^k$$

and, using the first expansion theorem (see [5]), we find

$$\left(R_{n,a}x^k\right)(0)=\frac{1}{n!}\langle n\rangle_k\langle n\rangle_{n-k}a^k.$$

Hence

$$\left(R_{n,a}x^k\right)(0)=\binom{n}{k}a^k$$

and

$$\sum_{k=0}^{n}\left(R_{n,a}x^k\right)(0)=(1+a)^n.$$

We prove by calculus that $R_{n,a}$ verifies the differential operator equation

$$DY''+(I+aD)Y'-anY=0,\tag{3}$$

where Y' and Y'' are the Pincherle derivatives, noticing the analogy with the differential equation

$$xy''+(1-x)y'+ny=0$$

verified by the Laguerre polynomials $L_n(x)$.

3. THE GENERATING FUNCTION

Let $\left(\Pi^*\right)^N$ be the ring of formal series with the elements $(T_0,T_1,T_2,\dots)$ and the operations

$$(T_0,T_1,T_2,\dots)+(S_0,S_1,S_2,\dots)=(T_0+S_0,T_1+S_1,T_2+S_2,\dots)$$

$$(T_0,T_1,T_2,\dots)\cdot(S_0,S_1,S_2,\dots)=(U_0,U_1,U_2,\dots),$$

where $U_n=\sum_{k=0}^{n}T_kS_{n-k},\ n=0,1,2,\dots$.

Let now $\varphi:\Pi^*\to\left(\Pi^*\right)^N$ be the injective unit morphism of rings given by $\varphi(T)=(T,0,0,\dots)$.

This injective morphism allows us to match the operator $T\in\Pi^*$ with its image $\varphi(T)$ from $\left(\Pi^*\right)^N$. If we note $X=(0,I,0,\dots)$, then we have $X^n=(\underbrace{0,0,\dots,0}_{n\text{ zeros}},I,0,\dots)$, where I is the identity operator.

With this notation, the element $f=(T_0,T_1,T_2,\dots)$ of $\left(\Pi^*\right)^N$ can be expressed by $f=\sum_{k=0}^{\infty}T_kX^k$, where $X^0=I$.

Let us consider now

$$R_{n,a}=\sum_{k=0}^{n}\frac{n!}{(k!)^2(n-k)!}(aD)^k,$$

hence

$$\sum_{n=0}^{\infty} g_n R_{n,a}X^n=\sum_{n=0}^{\infty} g_n\left(\sum_{k=0}^{n}\frac{n!}{(k!)^2(n-k)!}(aD)^k\right)X^n.$$

Using the identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k),$$

we find

$$\sum_{n=0}^{\infty} g_nR_{n,a}X^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n} g_n\frac{n!}{(k!)^2(n-k)!}(aD)^kX^n=$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_{n+k}\frac{(n+k)!}{(k!)^2 n!}(aD)^kX^{n+k}$$

and hence

$$\sum_{n=0}^{\infty} g_nR_{n,a}X^n=\sum_{n=0}^{\infty}\frac{X^n}{n!}\sum_{k=0}^{\infty} g_{n+k}\frac{(n+k)!}{(k!)^2}(aDX)^k.$$

Taking $g_n=\frac{1}{n!}$, we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}R_{n,a}X^n=\sum_{n=0}^{\infty}\frac{X^n}{n!}\sum_{k=0}^{\infty}\frac{(aDX)^k}{(k!)^2}$$

and so

$$\sum_{n=0}^{\infty}\frac{1}{n!}R_{n,a}X^n=e^X{}_0F_1(-1;1;\,aDX).$$

4. THE DELTA OPERATOR $Q_{n,a}=R_{n,a}-I$

Let us consider the delta operator $Q_{n,a}=R_{n,a}-I$ for $n\ge 1$. Its indicator is

$f(t)=\frac{1}{n!}\left(t^ne^{at}\right)^{(n)}e^{-at}-1$ (because of the Leibnitz formula).

The indicator of $Q'_{n,a}$ is

$$f'(t)=\frac{1}{n!}\left[\left(t^ne^{at}\right)^{(n+1)}e^{-at}-a\left(t^ne^{at}\right)^{(n)}e^{-at}\right],$$

hence

$$n!\,f'(t)=\frac{\left(t^ne^{at}\right)^{(n+1)}-a\left(t^ne^{at}\right)^{(n)}}{e^{at}}=$$

$$=\frac{\sum_{k=0}^{n+1}\binom{n+1}{k}\left(t^n\right)^{(n+1-k)}a^ke^{at}-a\sum_{k=0}^{n}\binom{n}{k}\left(t^n\right)^{(n-k)}a^ke^{at}}{e^{at}}$$

Because $\left(t^n\right)^{(n+1)}=0$, we have

$$n!\,f'(t)=\sum_{k=1}^{n+1}\binom{n+1}{k}\left(t^n\right)^{(n+1-k)}a^k-\sum_{k=0}^{n}\binom{n}{k}\left(t^n\right)^{(n-k)}a^{k+1}=$$

$$=\sum_{k=0}^{n}\left[\binom{n+1}{k+1}\left(t^n\right)^{(n-k)}a^{k+1}\right]-\sum_{k=0}^{n}\binom{n}{k}\left(t^n\right)^{(n-k)}a^{k+1}=$$

$$=\sum_{k=0}^{n}\left[\binom{n+1}{k+1}-\binom{n}{k}\right]\left(t^n\right)^{(n-k)}a^{k+1}=$$

$$=\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{n!}{k!}a^{k+1}t^k=\sum_{k=1}^{n}\binom{n}{k}\frac{n!}{(k-1)!}\frac{(at)^k}{t}=$$

$$=\frac{n}{t}\sum_{k=1}^{n}\binom{n}{k}\frac{(n-1)!}{(k-1)!}(at)^k=\frac{nl_n(-at)}{t};$$

$l_n(x)$ is the sequence of basic polynomials for the Laguerre delta operator $K=\frac{D}{D-I}$, and $l_n(x)=xe^xD^ne^{-x}e_{n-1}$, where $e_{n-1}=x^{n-1}$.

So

$$\frac{1}{f'(t)}=\frac{(n-1)!}{a\left(e^xD^ne^{-x}e_{n-1}\right)(-at)}.$$

If

$$\frac{1}{f'(t)} = \varphi_n(t),$$

we have

$$Q'^{-1}_{n,a} = \varphi_n(D).$$

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