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EXISTENCE AND APPROXIMATION OF POSITIVE FIXED POINTS OF NONEXPANSIVE MAPS

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1. INTRODUCTION

Throughout this paper E will be a real Banach space and $K \subset E$ *a cone*, i.e., a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$. Since we do not assume $K \cap (-K) = \{0\}$, the cone K can be, in particular, the entire space E. We shall denote by K^* the *dual cone*, i.e.,

$$K^* = \left\{x^* \in E^*; (x^*, x) \geq 0 \text{ for all } x \in K\right\}.$$

Also, by U and U_1 we shall denote open bounded subsets of E containing the origin; we shall assume that

$$\overline{U}_1 \subset U \subset E,$$

and we shall write K_U instead of $K \cap U$.

The following two fixed point theorems have been established in [8] by means of the continuation method, but without using the index theory. In the particular case when U and U_1 are two balls, $U = B_R(0)$ and $U_1 = B_r(0)$, $0 < r < R$, these results have been first obtained by K. Deimling [4] (see also [3] and [5] for related topics) by means of a different method. Although in [8] we have supposed that $K \cap (-K) = \{0\}$, the reader can easily see that such an assumption is not necessary.

THEOREM 1.1 [8]. *Let $f : \overline{K}_U \to E$ be α-condensing and suppose that the following conditions hold:*

(1.1) $\left(x^*, f(x)\right) \geq 0$ *for all* $x \in \overline{U} \cap \partial K$ *and* $x^* \in K^*$ *with* $\left(x^*, x\right) = 0$

(weak inwardness condition);

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$$f(x) \neq \lambda x \quad for \quad all \quad x \in K \cap \partial U \quad and \quad \lambda > 1. \tag{1.2}$$

Then f has a fixed point $x \in \overline{K}_U$.

Theorem 1.1 is a simple consequence of Theorem 3.1 in [8].

In particular, for $K = E$, condition (1.1) trivially holds and Theorem 1.1 reduces to the well-known continuation principle for α-condensing maps.

The next theorem is useful when $f(0) = 0$ and fixed points in $K \setminus \{0\}$ are of interest.

THEOREM 1.2 [8]. *Let* $f : \overline{K}_U \to E$ *be an* α*-condensing map satisfying* (1.1) *and* (1.2). *In addition, suppose*

$$x - f(x) \neq \lambda e \quad for \quad all \quad x \in K \cap \partial U_1 \quad and \quad \lambda > 0 \tag{1.3}$$

for some $e \in K \setminus \{0\}$. *Then f has a fixed point in* $K \cap (\overline{U} \setminus U_1)$.

For an example illustrating Theorem 1.2 we refer to [3, Example 20.1].

The aim of this paper is to obtain similar results for nonexpansive maps. Moreover, we shall get generalizations of the following continuation theorems for nonexpansive maps recently proved in [9]:

THEOREM 1.3 [9]. *Suppose E is uniformly convex and that, in addition, U is convex. Let* $f : \overline{U} \to E$ *be a nonexpansive map such that*

$$f(x) \neq \lambda x \quad for \quad all \quad x \in \partial U \quad and \quad \lambda > 1. \tag{1.4}$$

Then f has a fixed point in $\overline{U}$.

THEOREM 1.4 [9]. *Suppose E is a Hilbert space and* $f : \overline{U} \to E$ *is a nonexpansive map satisfying* (1.4) *(where U is not necessarily convex). Then f has a fixed point in* $\overline{U}$.

2. POSITIVE FIXED POINTS OF WEAKLY INWARD NONEXPANSIVE MAPS

THEOREM 2.1. *Suppose E is uniformly convex and that, in addition, U is convex. Let* $f : \overline{K}_U \to E$ *be a nonexpansive map satisfying* (1.1) *and* (1.2). *Then f has a fixed point in* $\overline{K}_U$.

Proof. For each $n \in \mathbf{N}, n \geq 2$, define the map

$$f_n : \overline{K}_U \to E, \quad f_n(x) = \left(1 - \frac{1}{n}\right) f(x). \tag{2.1}$$

Since f is nonexpansive, f_n is a contraction and, consequently, α-condensing. Moreover, since f satisfies (1.1) and (1.2), it easily follows that f_n also satisfies these conditions. Therefore, by Theorem 1.1, there exists a (unique) fixed point $x_n \in \overline{K}_U$ of f_n, that is,

$$\left(1 - \frac{1}{n}\right) f(x_n) = x_n. \tag{2.2}$$

Since any uniformly convex space is reflexive and $\overline{K}_U$ is convex bounded closed, there is a subsequence of (x_n) (also denoted by (x_n)) weakly convergent to some $x \in \overline{K}_U$. Further, $f(\overline{K}_U)$ being bounded, from (2.2) we obtain that

$$x_n - f(x_n) \to 0 \quad \text{strongly.}$$

Now the conclusion follows by

LEMMA 2.2 [1]. *Suppose E is uniformly convex. Let* $f : D \to E$ *be a nonexpansive map, where* $D \subset E$ *is a convex bounded closed set. If for a sequence* $(x_n) \subset D$ *one has* $x_n \to x$ *weakly and* $x_n - f(x_n) \to y$ *strongly, then* $x - f(x) = y$. ■

In Hilbert spaces, by (2.2) and the identity

$$2(a_n x_n - a_m x_m, x_n - x_m) = (a_n + a_m)|x_n - x_m|^2 + (a_n - a_m)\left(|x_n|^2 - |x_m|^2\right),$$

with $a_n = 1/(n-1)$, we can even prove (see [2] or [9]) that the entire sequence (x_n) is strongly convergent, without assuming the convexity of U. Thus, in Hilbert spaces, we additionally obtain an *approximation scheme* for a fixed point of f. More exactly, we have

THEOREM 2.3. *Suppose E is a Hilbert space. Let* $f : \overline{K}_U \to E$ *be a nonexpansive map satisfying* (1.1) *and* (1.2) *(where U is not necessarily convex). Then the sequence* $(x_n) \subset \overline{K}_U$ *given by* (2.2) *strongly converges to a fixed point of f.*

Remark. For $K = E$, Theorems 2.1 and 2.3 reduce to Theorems 1.3 and 1.4, respectively.

3. NONZERO FIXED POINTS

This section deals with the existence and approximation of fixed points in $K \setminus \{0\}$ of weakly inward nonexpansive maps which may have 0 as a fixed point.

THEOREM 3.1. *Suppose E is uniformly convex. In addition, assume that*

(3.1) $$0 \notin \operatorname{conv}(K \cap \partial B_1(0))$$

and that U is convex. Let $f: \overline{K}_U \to E$ *be a nonexpansive map satisfying* (1.1) *and* (1.2). *Also, suppose that there is* $e \in K \setminus \{0\}$ *such that*

(3.2) $$\overline{\{x - f(x); x \in K \cap \partial U_1\}} \cap \mathbf{R}_+ e = \emptyset.$$

Then f has a fixed point in $\overline{K}_U \setminus \{0\}$.

Proof. For each $n \in \mathbf{N}, n \geq 2$, the map f_n given by (2.1) satisfies (1.1), (1.2) and also (1.3) for n large enough, say $n \geq n_0$. Indeed, otherwise it would exist the sequences $(n_k) \subset \mathbf{N}, (x_n) \subset K \cap \partial U_1$ and $(\lambda_k) \subset \mathbf{R}_+^*$ such that $n_k \to \infty$ and

$$x_k - \left(1 - \frac{1}{n_k}\right) f(x_k) = \lambda_k e \text{ for all } k.$$

Clearly, (λ_k) is bounded and so we may suppose $\lambda_k \to \lambda_0$ for some $\lambda_0 \in \mathbf{R}_+$. It follows

$$x_k - f(x_k) \to \lambda_0 e,$$

which contradicts (3.2).

Therefore, according to Theorem 1.2, for each $n \geq n_0$, there exists $x_n \in K \cap (\overline{U} \setminus U_1)$ a fixed point of f_n. Further, as in the proof of Theorem 2.1, there is a subsequence of (x_n) weakly convergent to some $x \in \overline{K}_U$. Since $x_n \notin U_1$, by (3.1), we see that $x \neq 0$. Finally, by Lemma 2.2, we obtain $f(x) = x$. ∎

Remark. Condition (3.1) implies that K is normal, i.e.,

$$\inf\{|x + y|; x, y \in K \cap \partial B_1(0)\} > 0.$$

In Hilbert spaces we have a more precise result.

THEOREM 3.2. *Suppose E is a Hilbert space. Let* $f: \overline{K}_U \to E$ *be a nonexpansive map satisfying* (1.1), (1.2) *and* (3.2). *Then the sequence* $(x_n)_{n \geq n_0} \subset K \cap (\overline{U} \setminus U_1)$ *given by* (2.2) *strongly converges to a fixed point* $x \in K \cap (\overline{U} \setminus U_1)$ *of f.*

4. OPERATOR INCLUSIONS WITH HYPERACCRETIVE MAPS

A map $A: E \to 2^E$ is said to be *hyperaccretive* provided that the following two conditions hold:

$$(u - v, x - y)_+ \geq 0 \text{ for all } x, y \in D(A), \quad u \in A(x) \text{ and } v \in A(y),$$
$$(A + I)(E) = E,$$

where $(x, y)_+ = |y| \lim_{t \to 0^+} t^{-1}(|y + tx| - |y|)$.

For a hyperaccretive map A one considers the nonexpansive map

(4.1) $$f: E \to E, f(x) = (A + I)^{-1}(x).$$

In this section we deal with the solvability of the inclusion $0 \in A(x)$, or, equivalently, of the equation $(A + I)^{-1}(x) = x$, where A is a hyperaccretive map. The results are direct consequences of the theorems of Sections 2 and 3.

THEOREM 4.1. *Suppose E is uniformly convex and* $A: E \to 2^E$ *is a hyperaccretive map. In addition, assume that*

(4.2) $$A(A + I)^{-1}(\partial K) \subset -K,$$

(4.3) $$(u, x)_+ \geq 0 \text{ for all } x \in K \text{ with } |x| > R \text{ and } u \in A(x)$$

(*coerciveness with respect to zero*),

for some $R > 0$. *Then there exists* $x \in K$ *with* $|x| \leq R$ *and* $0 \in A(x)$.

Proof. Take $U = B_R(0)$ and f given by (4.1). Then check that (4.2) implies (1.1), while (4.3) implies (1.2). Thus the conclusion follows by Theorem 2.1. ∎

Remark. If instead of (4.3) we require that f is *coercive on K*, i.e.,

(4.4) $$(u, x)_+ / |x| \to \infty \text{ as } x \in K \text{ and } |x| \to \infty$$

for each selection $u \in A(x)$, and instead of (4.2) that

(4.5) $$A(A + I)^{-1}(K) \subset -K,$$

then for each $h \in K$ there exists $x \in K$ with $h \in A(x)$ (apply Theorem 4.1 to $A - h$).

THEOREM 4.2. *Suppose E is uniformly convex and K satisfies* (3.1). *Let* $A: E \to 2^E$ *be a hyperaccretive map satisfying* (4.2), (4.3) *and*

(4.6) $$\overline{A(A + I)^{-1}(K \cap \partial B_r(0))} \cap \mathbf{R}_+ e = \emptyset$$

for some $e \in K \setminus \{0\}$ *and* $r \in]0, R[$. *Then there exists* $x \in K \setminus \{0\}$ *with* $|x| \leq R$ *and* $0 \in A(x)$.

Proof. Apply Theorem 3.1 to $U = B_R(0), U_1 = B_r(0)$ and f given by (4.1). ∎

THEOREM 4.3. *Suppose E is a Hilbert space. Let* $A: E \to 2^E$ *be a hyperaccretive map satisfying* (4.2), (4.3) *and* (4.6). *Then the sequence* $(x_n)_{n \ge n_0} \subset K$, $r \le |x_n| \le R$,

$$\left(1 - \frac{1}{n}\right)(A + I)^{-1}(x_n) = x_n,$$

strongly converges to a solution $x \in K$ *of* $0 \in A(x)$, *and* $r \le |x| \le R$.

For other applications of the continuation principles to the theory of nonlinear maps of monotone type we refer to [6] and [7].

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