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A GENERAL FUNCTIONAL INEQUALITY AND ITS APPLICATIONS

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1. INTRODUCTION

In [10] we studied the inequalities involving Picard operators. The main result in that paper is the following abstract Gronwall theorem:

THEOREM 1. *Let* $(X,d,\leq)$ *be an ordered metric space. Let* $A:X\to X$ *be such that:*

(i) A is monotonically increasing;

(ii) A is a Picard operator $\left(F_A=\{x_A^*\}\right)$.

Then:

(a) $x\leq A(x)$ *implies* $x\leq x_A^*$;
(b) $x\geq A(x)$ *implies* $x\geq x_A^*$.

From this abstract there results a large class of integral inequalities (Gronwall, Bellmann, Giuliano, Harlamov, Willet, Wong, Beesack, Wendorff, etc.) and some abstract inequalities (see, for instance, [3] and [10]) follow.

The present paper deals with a new abstract functional inequality. Some applications to the integral inequality, which do not involve Picard operators, are given.

2. FUNCTIONAL INEQUALITIES

Our main result is the following

THEOREM 2. *Let* $F:[a,b]\times C([a,b],R_+)\to R_+$ *be a functional. We suppose that:*

(i) $F(\cdot,x)\in C^1([a,b],R_+)$, *for all* $x\in C([a,b],R_+)$;

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(ii) *there exists* $\phi \in C([a,b]\times R_+)$ *such that*

$$\left(F(\cdot,x)\right)'(t)\Big|_{x=x(t)} = \phi(t,x(t)),$$

for all $t \in [a,b]$;

(iii) *the function* $\phi(t,\cdot)$ *is monotonically increasing for all* $t \in [a,b]$;

(iv) $(F(\cdot,x))'(t) \le \phi(t,x(t))$ *for all* $t \in [a,b]$, *and all monotonically increasing* $x \in C([a,b],R_+)$;

(v) *there exists* $\alpha \in R_+$, *such that* $F(a,x) = \alpha$, *for all* $x \in C([a,b],R_+)$.

Let:

(a) $x \in C([a,b],R_+)$ *be a solution of the inequality*

$$x(t) \le F(t,x), \ \text{for all } t \in [a,b];$$

(b) y *be the maximal solution of the following Cauchy problem*

$$y'(t) = \phi(t,y(t)), \ \forall t \in [a,b],$$
$$y(a) = \alpha.$$

Then

$$x \le y.$$

Proof. Let $u(t) = F(t,x)$. From (iii) we have

$$u'(t) = (F(\cdot,x))'(t) \le (F(\cdot,z))'(t)\Big|_{z=x(t)} \le (F(\cdot,z))'(t)\Big|_{z=u(t)}.$$

Hence

$$u'(t) \le \phi(t,u(t)),$$
$$u(a) = \alpha.$$

By the theorem of differential inequalities ([3], [9], [11]) it follows that

$$u \le y, \ \text{i.e.,}\ x \le y.$$

Remark 1. We can take $[a,b[$ or $[a,+\infty[$, instead of [a, b], in Theorem 2.

Remark 2. If

$$(F(\cdot,x))'(t)\Big|_{x=x(t)} \le \phi(t,x(t)), \ t \ge a,$$

then the conclusion of Theorem 2 follows.

Remark 3. If

$$(F(\cdot,x))'(t) \le \phi(t,x(t),x(g(t))), \ t \ge a$$

for the solutions x of $x \le F(t,x)$, where $g(t) \ge a$, for all $t \ge a$, and for the following Cauchy problem

$$y'(t) = \phi(t,y)$$
$$y(a) = \alpha$$

we have a theorem of differential inequalities, then the conclusion of Theorem 2 follows.

3. SOME INTEGRAL INEQUALITIES

From Theorem 2 and Remarks 1, 2 and 3 on this theorem, we have

THEOREM 3 (see [6], [8]). *Let* $\alpha \in R_+, p$ *and* $q \in C([a,+\infty[,R_+)$. *If* $x \in C([a,+\infty[,R_+)$ *is such that*

$$x(t) \le \alpha^2 + 2\int_a^t \left[p(s)x(s) + q(s)\sqrt{x(s)}\right]\mathrm{d}s,$$

for all $t \ge a$, *then*

$$\sqrt{x(t)} \le \left[\alpha + \int_a^t q(s)\mathrm{d}s\right]\exp\int_a^t p(s)\mathrm{d}s.$$

Proof. We have

$$F(t,x) := \alpha^2 + 2\int_a^t \left[p(s)x(s) + q(s)\sqrt{x(s)}\right]\mathrm{d}s,$$

and

$$\phi(t,y) = 2p(t)y(t) + 2q(t)\sqrt{y(t)}.$$

The Cauchy problem

$$y'(t) = 2p(t)y(t) + 2q(t)\sqrt{y(t)}, \ t \ge a$$
$$y(a) = \alpha^2$$

has a unique solution

$$x(t) = \left\{\left[\alpha + \int_a^t q(s)\mathrm{d}s\right]\exp\int_a^t p(s)\mathrm{d}s\right\}^2.$$

THEOREM 4 (see [7]). *Let* $\alpha, \beta \in R_+$ *and* $p, q \in C([a, +\infty[, R_+)$. *If* $x \in C([a, +\infty[, R_+)$ *is a solution of the inequality*

$$x(t) \le \left[\alpha + \int_a^t p(s)x(s)\mathrm{d}s\right]\left[\beta + \int_a^t q(s)x(s)\mathrm{d}s\right],$$

for all $t \in [a, +\infty[$, *then*

$$x \le y,$$

where y is a unique solution of the following Cauchy problem

$$y'(t) = [\alpha q(t) + \beta p(t)]y(t) + \left[p(t)\int_a^t q(s)\mathrm{d}s + q(t)\int_a^t p(s)\mathrm{d}s\right]y^2(t)$$

$$y(a) = \alpha.$$

Proof. We have

$$F(t, x) = \left[\alpha + \int_a^t p(s)x(s)\mathrm{d}s\right]\left[\beta + \int_a^t q(s)x(s)\mathrm{d}s\right]$$

and

$$\phi(t, y(t)) = (F(\cdot, x))'(t)\Big|_{x=y(t)} = [\alpha q(t) + \beta p(t)]y(t) + \left[p(t)\int_a^t q(s)\mathrm{d}s + q(t)\int_a^t p(s)\mathrm{d}s\right]y^2(t).$$

It is easy to observe that F and ϕ satisfy the conditions of Theorem 2. Applying Theorem 2 with the above F and ϕ, we obtain the conclusion of Theorem 4.

THEOREM 5 (see [1]). *Let* $a, \alpha \in R_+$, $p, q \in C([a, +\infty[, R_+)$ *and* $g \in C([a, +\infty[, [a, +\infty[)$. *Let* $x \in C([a, +\infty[, R_+)$ *be a solution of the inequality*

$$x(t) \le \alpha + \int_a^t p(s)x(s)\mathrm{d}s + \int_a^t q(s)x(g(s))\mathrm{d}s.$$

If g is monotonically increasing and $g(t) \le t$, *for all* $t \in [a, +\infty[$, *then*

$$x \le y,$$

where y is a unique solution of the following Cauchy problem

$$y'(t) = p(t)y(t) + q(t)y(g(t)), \quad t \ge a$$

$$y(a) = \alpha.$$

Proof. We have

$$F(t, x) = \alpha + \int_a^t p(s)x(s)\mathrm{d}s + \int_a^t q(s)x(g(s))\mathrm{d}s$$

and

$$\phi(t, y(t)) = p(t)y(t) + q(t)y(g(t)).$$

The proof follows from Remark 3.

Remark 4. Theorem 1 and Theorem 2 improve, unify and extend the results of Gronwall (1919), Chaplighin (1919), Bellmann (1943), Giuliano (1946), Harlamov (1955), Willet-Wong (1965), Beesack (1969, 1975), Wendorff, Li, Bihari, Turinici, Lungu, Lakshmikantham, Leela, Young, Bainov, Ou-Iang, Defermos, Pachpatte, and of many others (see [2], [3], [10], [12]).

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