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ON A MEDIAN FOR ONE SPECIAL SPACE

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The following problem is formulated in [1]. Let R be a vector plane on the field of real numbers with norm $\|x\| = |x^1| + |x^2|$; $M \subset R^2$ is a polygon, topologically equal with a Euclidean circle, and every side of it is parallel (see Figure) to one axis of coordinate system of $R^2, S = \{x_1, x_2, \ldots, x_m\} \subset M$ is a set of points having, respectively, the positive weights $p(x_1), p(x_2), \ldots, p(x_m)$.

It is required to find a median in M, i.e., such a point $x_0 \in M$ that minimizes the function

$$f(x) = \sum_{x \in M} p(x_i) d(x, x_i),$$

where $d(x, x_i)$ represents a distance between points x and x_i calculated following a curve of a minimal length in space $M \subset R^2$ that connects the points.

This problem is solved in [1] by using a complicated algorithm having, however, the advantage of a linear complexity.

In this paper the other algorithm for the indicated problem is offered; it is based on d-convexity theory and follows from the algorithms developed for finding a median in [2]. The maximal parallels to axes segments (by inclusion in M) through any point of local nonconvexity [4] (see point x_m on the Figure) of a polygon M and set S are drawn. So a polygon M is transformed to figure that is divided into parallelograms with their sides forming a grid in M; the last is denoted by graph $G = (X, V)$, where X is the set of vertices (nodes) and V is the set of edges. The points $x_1, x_2, \ldots, x_m$ represent a subset of vertices of G. Denote the other vertices of G by $x_{m+1}, x_{m+2}, \ldots, x_n$. Hence, we obtain the new set of points X of a polygon M.

Now we assign a new positive weight to every point $x \in X$. If $x_i \in X \setminus S$, then we put to x_i the weight $q(x_i) = 1, (i = m+1, \ldots, n)$; if $x_i \in S$, then $q(x_i) = p(x_i) + 1, (i = 1, \ldots, m)$. The new problem is formulated in the following way: to find such a vertex $x_0 \in X$ of graph G that minimizes the function

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$$\varphi(x) = \sum_{x \in X} q(x_i) d(x, x_i).$$

According to norm $\|x\|$ in R^2 and to the results of [2], $d(x, x_i)$ of this new problem is the same distance that participates in the definition of the function $f(x)$.

Applying the reasoning that was presented in [2], we obtain that graph G satisfies the conditions indicated in this paper, which permit us to solve a new formulated problem by the same algorithm, since it is not necessary to know distances $d(x, x_i)$, $i = 1, 2, \ldots, n$.

The essence of this algorithm is the following.

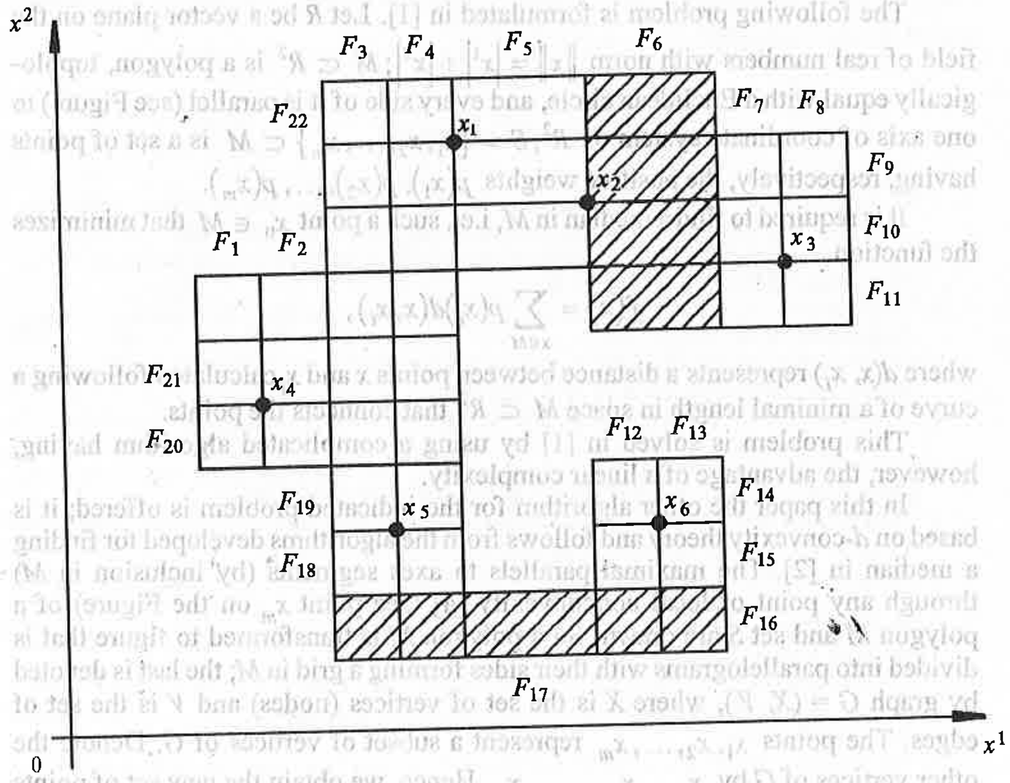

Fig. 1

Let s be the number of horizontal and vertical strips in which polygon M is divided. Denote these strips by $F_1, F_2, \ldots, F_j, \ldots, F_s$ (on Fig. 1 the examples of strips are indicated by horizontal sides and $s = 22$).

Put in correspondence to any vertex of graph G a sequence from 0 and 1 of length s according to the following conditions: 1) to point x_1 it is put in correspondence sequence $\varepsilon^1 = (\varepsilon_1^1, \varepsilon_2^1, \ldots, \varepsilon_j^1, \ldots, \varepsilon_s^1)$, where $\varepsilon_j^1 = 0$, $j = 1, 2, \ldots, s$;

2) to point x_i, $i = 1, 2, \ldots, n$ it is put a sequence $\varepsilon^i = (\varepsilon_1^i, \varepsilon_2^i, \ldots, \varepsilon_j^i, \ldots, \varepsilon_s^i)$, where $\varepsilon_j^i = 0$, if an arbitrary chain of a graph G connecting the vertices x_1 and x_i intersects a piece F_j an even number of times, and $\varepsilon_j^i = 1$, otherwise. So we construct a function $\alpha : X \to \{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^i, \ldots, \varepsilon^n\}$ which, as it follows from [2], is one-to-one.

Further, construct a new sequence $r = (r^1, r^2, \ldots, r^j, \ldots, r^s)$ of elements 0 and 1 in correspondence to the conditions:

1) $r_j = 0$, if

$$\sum_{i=1}^{n} q(x_i)(1 - \varepsilon_j^i) > \frac{1}{2} \sum_{i=1}^{n} q(x_i);$$

2) $r_j = 1$, if

$$\sum_{i=1}^{n} q(x_i)(1 - \varepsilon_j^i) < \frac{1}{2} \sum_{i=1}^{n} q(x_i);$$

3) $r_j = 0$ or $r_j = 1$, if

$$\sum_{i=1}^{n} q(x_i)(1 - \varepsilon_j^i) = \frac{1}{2} \sum_{i=1}^{n} q(x_i).$$

In virtue of [2], there exists an index i_0, $1 \le i_0 \le n$ such that $r = \varepsilon^{i_0}$. If x_0 is such a vertex of a graph G for which we have $\alpha(x_0) = \varepsilon^0 = r$, then $x_0 = \alpha^{-1}(\varepsilon_0)$ minimizes the function $\varphi(x)$. Moreover the following statement holds:

THEOREM 1. *The vertices of graph $G = (X, U)$ that minimize the function $\varphi(x)$ are the vertices which give the minimal values for the $f(x)$.*

Proof. Consider the function

$$\varphi_\varepsilon(x) = \sum_{i=1}^{n} q^s(x_i) d(x, x_i),$$

where ε is an arbitrary positive number and $q^s(x_i) = p(x_i)$ if $x_i \in S$ and $q^s(x_i) = p(x_i) + \varepsilon$ if $x_i \in X \setminus S$.

It is obvious that the function $\varphi(x)$ satisfies conditions 1)–3) if and only if the function $\varphi_\varepsilon(x)$ also satisfies these conditions. Therefore, by virtue of [2], the vertices of a graph $G = (X, U)$, that minimize functions $\varphi(x)$, $\varphi_\varepsilon(x)$, are the same.

Now we extend the function $\varphi_\varepsilon(x)$ that is defined on the set X onto the whole polygon $M \subset R^2$, preserving the same notation for it. Evidently, it is easy to do this operation by virtue of definitions for distance $d(x, x_i)$ between points x, x_i and norm $\|x\|$ of the space R^2. In the same way transform the function $f(x)$ by

$$f(x) = \sum_{i=1}^{n} p(x_i) d(x, x_i),$$

where $p(x_i) = 0$ for $i = m+1, m+2, \dots, n$, that is, for $x \in X \setminus S$. Observe that for any $x \in M$ we have the relations $\varphi_\varepsilon(x) = f(x) + n \cdot \varepsilon > 0$. Hence, if $\varepsilon \to 0$ for any $x \in M$, we obtain $(\varphi_\varepsilon(x) - f(x)) \to 0$. It follows that point $x_0 \in M$ minimizing the function $\varphi_\varepsilon(x)$ will minimize the function $f(x)$ (generally the inverse statement does not hold). The theorem is proved.

This theorem permits us to reduce the solving of the initial problem to finding the median for function $\varphi_\varepsilon(x)$. So, we obtain that in case $\varepsilon = 1$ the algorithm for finding the median for the function $\varphi(x)$ is the same as that for finding the median for the function $f(x)$.

Note. As it is proved in [3], the set of arguments minimizing the function $\varphi(x)$ is d-convex. Hence, applying in addition the results of [2], we obtain that the set of medians of the function $\varphi(x)$ may represent one of the following possibilities: 1) the case when we have a single sequence r, respectively, one point $x_0 = \alpha^{-1}(r)$; 2) the case when we have two sequences r_1 and r_2, respectively, one segment $u = [x^1 = \alpha^{-1}(r_1),\ x^2 = \alpha^{-1}(r_2)] \subset U$ that is parallel to one of the axes; 3) the case when we have four sequences r_1, r_2, r_3, r_4, obtaining, respectively, one parallelogram that divides the polygon $M \subset R^2$ that has vertices $x^1 = \alpha^{-1}(r_1)$, $x^2 = \alpha^{-1}(r_2), x^3 = \alpha^{-1}(r_3), x^4 = \alpha^{-1}(r_4)$.

One can prove that this property remains valid for the function $f(x)$, but in this case, if M_φ and M_f are the sets of the respective median for $\varphi, f(x)$, then $M_\varphi \subset M_f$.

Direct realization of the expounded method by one algorithm gives the possibility to obtain the complexity $o(n^2)$, where n is the number of strips, and it is equal to the sum compiled from the number of given points m and the number of the edges of a polygon k. This complexity is determined by the mode of representation of a grid obtained as a result of polygon division.

Indeed, further calculations may be reduced to finding the median of two trees: one that is determined by the horizontal strips and the other determined by the vertical strips. To every strip it corresponds an edge of a tree. Two edges have one common vertex if the respective strips have a common border. The weights of this vertex equal the sum of weights of vertices on the grid that belong to this border.

For example, for the polygon pictured in Fig. 1 the horizontal tree is H_0 and the vertical tree is H_v (Fig. 2).

The finding of the median for every tree may be executed with the optimal (linear) complexity [2]. Information about the median of every tree determines the

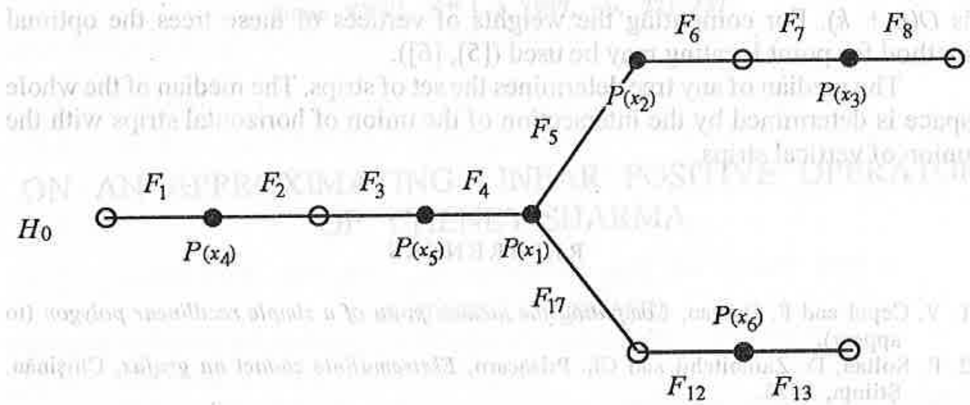

Fig. 2a

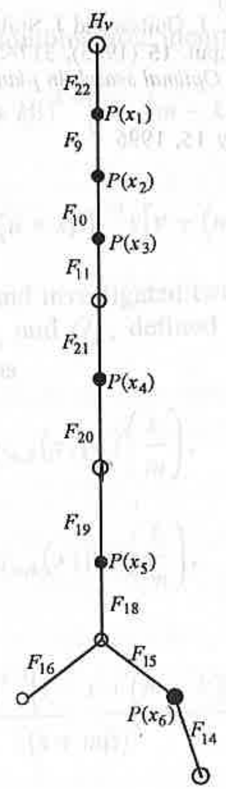

Fig. 2b

median of a polygon. In order to obtain the algorithm with optimal complexity for computing a median, it is sufficient to obtain the trees H_0 and H_1 with the optimal complexity.

It is possible to do this without the complete description of a grid obtained from the initial polygon. It is obvious that the number of edges of constructed trees is $O(n + k)$. For computing the weights of vertices of these trees the optimal method for point locating may be used ([5], [6]).

The median of any tree determines the set of strips. The median of the whole space is determined by the intersection of the union of horizontal strips with the union of vertical strips.

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