REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome XXVI, N^{os} 1–2, 1997, pp. 19–27

ON APPROXIMATING THE EIGENVALUES AND EIGENVECTORS OF LINEAR CONTINUOUS OPERATORS

EMIL CĂTINAŞ, ION PĂVĂLOIU

1. INTRODUCTION

One of the difficulties that appear when solving numerically the operatorial equations by iterative methods having high convergence orders is that at each iteration step there must be solved a linear operatorial equation, or, in some cases, even more than one [2].

In a series of papers ([6], [7], [9], [14]) there has been proposed the elimination of this inconvenience by the simultaneous generation of two sequences: a sequence approximating the solution, and a sequence of linear operators approximating the inverses of the operators appearing at each step in the linear equation.

In the present note we shall study this approach when the solutions of the equation yield eigenvalues and eigenvectors of a linear operator. It is well known that all the derivatives of an order higher than three of the attached operator are the null multilinear operators. The convergence results may be also applied to polynomial operator equations of the second degree.

The r convergence order of the sequence approximating the solution is proved to be 2.

The operatorial equation which leads us to the approximation of the eigenvalues and eigenvectors of a linear continuous operator can be constructed in the following way (see [5]).

Let V be a Banach space over the field $\mathbf{K}$ (where $\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{R}$); denote by $\mathcal{L}(V)$ the set of linear continuous operators acting from V into V and let $A \in \mathcal{L}(V)$. The scalar $\lambda \in \mathbf{K}$ is an eigenvalue of A iff the equation

$$Av - \lambda v = \theta \tag{1.1}$$

has at least a solution $v^* \neq \theta$, called an eigenvector of A corresponding to λ, where θ is the null element of V.

1991 AMS Subject Classification: 65J20, 65H17.

For the determination of an eigenpair (v, λ) we also consider another equation

$$G(v) - 1 = 0, \tag{1.2}$$

where $G : V \to \mathbf{K}$ is a polynomial functional of the second degree (a norming function) for which $G(0) \neq 1$.

Remark. The functional G may also be taken as a linear continuous functional, but then dim Ker $G = n-1$ for the finite dimensional case and infinite otherwise, so there exist eigenvectors which do not fulfil equation (1.2).

Denote the Banach space $X = V \times \mathbf{K}$ and for $x = \begin{pmatrix} v \\ \lambda \end{pmatrix}$, $v \in V$ and $\lambda \in \mathbf{K}$, define

$$\|x\| = \max\{\|v\|, |\lambda|\}.$$

Considering the operator $F : X \to X$ given by

$$F(x) = \begin{pmatrix} Av - \lambda v \\ G(v) - 1 \end{pmatrix} = \begin{pmatrix} (A - \lambda I)v \\ G(v) - 1 \end{pmatrix},$$

and denoting by $\bar{\theta} = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$ the null element of X, then the operatorial equation for which a solution yields an eigenpair of A is

$$F(x) = \bar{\theta}. \tag{1.3}$$

It is known that the Fréchet derivatives of F are (see [5])

$$F'(x_0)h = \begin{pmatrix} A - \lambda_0 I & -v_0 \\ G'(v_0) & 0 \end{pmatrix} \begin{pmatrix} u \\ \alpha \end{pmatrix} = \begin{pmatrix} Au - \lambda_0 u - \alpha v_0 \\ G'(v_0)u \end{pmatrix},$$

and

$$F''(x_0)hk = \begin{pmatrix} -\alpha w - \beta u \\ G''uw \end{pmatrix},$$

where $x_0 = \begin{pmatrix} v_0 \\ \lambda_0 \end{pmatrix}$, $h = \begin{pmatrix} u \\ \alpha \end{pmatrix}$, $k = \begin{pmatrix} w \\ \beta \end{pmatrix} \in X$.

It is obvious that $F^{(i)}(x_0)h_1 \dots h_i = \bar{\theta}$, for all $i \geq 3$ and $x_0, h_1, \dots, h_i \in X$, and also that $\|F''\| = \max\{2, \|G''\|\}$.

For such an operator the following identity holds

$$F(y) = F(x) + F'(x)(y - x) + F''(x)(y - x)^2, \text{ for all } x, y \in X, \tag{1.4}$$

where the bilinear operator F'' does not depend on x.

We shall consider two sequences $(x_k)_{k\geq 0}$, $(\Gamma_k)_{k\geq 0}$ having elements from X, respectively from $\mathcal{L}(X)$, using the following iterative method

$$\begin{aligned} x_{k+1} &= x_k - \Gamma_k F(x_k) \\ \Gamma_{k+1} &= \Gamma_k\big(2I - F'(x_{k+1})\Gamma_k\big), \quad k = 0, 1, \dots, \end{aligned} \tag{1.5}$$

where $x_0 \in X$, $\Gamma_0 \in \mathcal{L}(X)$ and I is the identity operator of $\mathcal{L}(X)$.

We call (1.5) the combined Newton method. In the following sections we shall study its convergence when F is a polynomial operator of the second degree and we shall apply it for the approximation of the eigenvalues and eigenvectors of matrices.

2. THE CONVERGENCE OF THE COMBINED NEWTON METHOD

For the study of the convergence of method (1.5) we shall use the following lemma:

LEMMA. *If the sequences $(\delta_k)_{k\geq 0}$ and $(\rho_k)_{k\geq 0}$ of real positive numbers satisfy*

$$\delta_{k+1} \leq (\delta_k + 2\rho_k)^2$$

$$\rho_{k+1} \leq \rho_k\delta_k + \rho_k^2, \quad k = 0, 1, \dots$$

where $\max\{\delta_0, \rho_0\} \leq \frac{1}{9}d$ *for some* $0<d<1$, *then the following relations hold:*

$$\delta_k \leq \frac{1}{9}d^{2^k},$$

$$\rho_k \leq \frac{1}{9}d^{2^k}, \quad k = 0, 1, \dots.$$

The proof of this Lemma is immediately obtained by induction.

Let $x_0 \in X$, $\Gamma_0 \in \mathcal{L}(X)$ and $S = \{x \in X \mid \|x - x_0\| \leq r\}$, $r > 0$. Suppose that we have the estimation

$$\|F''(x)\| \leq K \text{ for all } x \in S.$$

The following theorem holds:

THEOREM. *If x_0, Γ_0, r and F satisfy the following conditions*

a) *there exists* $F'(x_0)^{-1}$ *and* $b_0 = \|F'(x_0)^{-1}\|$,

b) $q = Kb_0 r < 1$,

c) $\max\{\delta_0, \rho_0\} \leq \frac{1}{9} d$, *for some* $0<d<1$, *where* $\delta_0 = \|I - F'(x_0)\Gamma_0\|$, $\rho_0 = \frac{50}{81} b^2 K \|F(x_0)\|$ *and* $b = \frac{b_0}{1-q}$,

d) $\frac{d}{5K(1-q)} \leq r$,

then the following properties hold:

1) *the sequences* $(x_k)_{k\geq 0}$, $(\Gamma_k)_{k\geq 0}$ *generated by* (1.5) *converge, and* $x_k \in S$ *for all* $k \geq 0$;

2) *denoting* $x^* = \lim_{k\to\infty} x_k$ *and* $\Gamma^* = \lim_{k\to\infty} \Gamma_k$, *then* $F(x^*) = \theta$ *and* $\Gamma^* = F'(x^*)^{-1}$;

3) *the following relations are true:*

$$\|x^* - x_k\| \leq \frac{d^{2^k}}{5bK\left(1 - d^{2^k}\right)},\quad k = 0, 1, \ldots;$$

$$\|\Gamma^* - \Gamma_k\| \leq \frac{d^{2^k}}{3\left(1 - d^{2^k}\right)},\quad k = 0, 1, \ldots.$$

Proof. At first we shall show that for any $x \in S$ there exists $F'(x)^{-1}$ and $\|F'(x)^{-1}\| \leq b$.

Indeed, under the above assumptions, it easily follows that

$$\|I - F'(x_0)^{-1} F'(x)\| \leq K b_0 r = q < 1.$$

Applying the Banach Lemma, we get

$$\|F'(x)^{-1}\| \leq \frac{b_0}{1-q} = b.$$

Taking into account a) and b), it follows that

$$(2.1)\quad \|\Gamma_0\| \leq \|F'(x_0)^{-1}\|\left(\|F'(x_0)\Gamma_0 - I\| + 1\right) \leq b_0(1 + \delta_0) \leq \frac{10}{9} b_0 \leq \frac{10}{9} b,$$

which, together with the first relation from (1.5), for $k = 0$, implies

$$\|x_1 - x_0\| \leq \|\Gamma_0\| \|F(x_0)\| \leq \frac{10 \cdot 81 bd}{9 \cdot 9 \cdot 50 b^2 K} < \frac{d}{5bK(1-d)} \leq r,$$

i.e., $x_1 \in S$.

Denote $\rho_1 = \frac{50}{81} b^2 K \|F(x_1)\|$ and $\delta_1 = \|I - F'(x_1)\Gamma_1\|$. An elementary reasoning shows that

$$\rho_1 \leq \rho_0^2 + \delta_0 \rho_0$$
$$\delta_1 \leq (\delta_0 + 2\rho_0)^2,$$

whence, by the above Lemma and by relation c), we infer that

$$\rho_1 \leq \frac{1}{9} d^2$$
$$\delta_1 \leq \frac{1}{9} d^2.$$

Suppose now that the following relations hold:

α) $x_1, \ldots, x_k \in S$;

β) $\delta_i = \|I - F'(x_i)\Gamma_i\| \leq \frac{1}{9} d^{2^i}$, $\rho_i = \frac{50}{81} b^2 K \|F(x_i)\| \leq \frac{1}{9} d^{2^i}$, $i = \overline{0, k}$;

γ) $\|x_i - x_{i-1}\| \leq \frac{d^{2^i}}{5bK}$, $i = \overline{1, k}$.

We shall show they also hold for $k+1$.

The inequality

$$\|\Gamma_k\| \leq \frac{10}{9} b$$

is proved similarly with (2.2). Using also the second relation from (1.5), we get

$$\|x_{k+1} - x_k\| \leq \|\Gamma_k\| \|F(x_k)\| \leq \frac{d^{2^k}}{5bK},$$

i.e., the relation γ) for $i = k+1$.

Further,

$$\|x_{k+1} - x_0\| \leq \sum_{i=0}^{k} \|x_{i+1} - x_i\| \leq \frac{1}{56K} \sum_{i=0}^{k} d^{2^k} < \frac{d}{5Kb(1-d)} \leq r,$$

whence it follows that $x_{k+1} \in S$.

Denoting $\rho_{k+1} = \frac{50}{81} b^2 K \|F(x_{k+1})\|$ and $\delta_{k+1} = \|I - F'(x_{k+1})\Gamma_{k+1}\|$, then

$$\rho_{k+1} \leq \rho_k^2 + \rho_k \delta_k$$
$$\delta_{k+1} \leq (\delta_k + \rho_k)^2,$$

whence, by our Lemma, we get

$$\rho_{k+1} \le \frac{1}{9} d^{2^{k+1}}$$

$$\delta_{k+1} \le \frac{1}{9} d^{2^{k+1}},$$

i.e., the relation β) for $i = k+1$.

It is obvious that $x_i \in S$ for all $i = 1, 2, \ldots$ and the relations β) and γ) hold for all $k \in \mathbb{N}$.

From the inequalities

$$\left\| x_{k+m} - x_k \right\| \le \sum_{i=k}^{k+m-1} \left\| x_{i+1} - x_i \right\| \le \frac{d^{2^k}}{5Kb\left(1 - d^{2^k}\right)}, \quad m = 1, 2, \ldots$$

it follows that the sequence $(x_k)_{k\ge 0}$ converges. Denoting $x^* = \lim_{k\to\infty} x_k$, then from the last relation for $m \to \infty$ we get

$$\left\| x^* - x_k \right\| \le \frac{d^{2^k}}{5Kb\left(1 - d^{2^k}\right)}, \quad k = 0, 1, \ldots .$$

From the second relation of (1.5) it follows that

$$\left\| \Gamma_{k+1} - \Gamma_k \right\| = \left\| I - F'(x_{k+1})\Gamma_k \right\| \le \left\| I - F'(x_k)\Gamma_k \right\| + \left\| F'(x_k) - F'(x_{k+1}) \right\| \left\| \Gamma_k \right\| \le$$

$$\le \delta_k + \left\| \Gamma_k \right\|^2 K \left\| F(x_k) \right\| \le \delta_k + 2\rho_k \le \frac{1}{3} d^{2^k}.$$

The previous inequality implies that the sequence $(\Gamma_k)_{k\ge 0}$ converges and the relations 2) and the second inequality in 3) hold. ∎

3. THE APPROXIMATION OF EIGENVALUES AND EIGENVECTORS OF MATRICES

In the following we shall apply the studied method for the approximation of the eigenvalues and eigenvectors of complex matrices.

Let $A = (a_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{K})$ be a square matrix with the elements $a_{ij} \in \mathbb{K}$. Consider $V = \mathbb{K}^n$ and $X = \mathbb{K}^{n+1}$. In this case equation (1.3) is written

$$F_i(x) = F_i\left(x^{(1)}, \ldots, x^{(n+1)}\right) = 0, \quad i = \overline{1, n+1},$$

where

$$F_i(x) = a_{i1}x^{(1)} + \ldots + a_{i,i-1}x^{(i-1)} + \left(a_{ii} - x^{(n+1)}\right)x^{(i)} + a_{i,i+1}x^{(i+1)} + \ldots + a_{in}x^{(n)}, \; i = \overline{1, n}.$$

For the norming function we can take $G = F_{n+1}$, where

$$F_{n+1}(x) = \frac{1}{2}\sum_{i=1}^{n}\left(x^{(i)}\right)^2 - 1. \tag{3.1}$$

The first- and the second-order derivatives of F are given by

$$F'(x)h = \begin{pmatrix} a_{11} - x^{(n+1)} & a_{12} & \cdots & a_{1n} & -x^{(1)} \\ a_{21} & a_{22} - x^{(n+1)} & \cdots & a_{2n} & -x^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x^{(n+1)} & -x^{(n)} \\ x^{(1)} & x^{(2)} & \cdots & x^{(n)} & 0 \end{pmatrix} \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ \vdots \\ h^{(n)} \\ h^{(n+1)} \end{pmatrix},$$

and

$$F''(x)hk = \begin{pmatrix} -k^{(n+1)} & 0 & \cdots & 0 & -k^{(1)} \\ 0 & -k^{(n+1)} & \cdots & 0 & -k^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -k^{(n+1)} & -k^{(n)} \\ k^{(1)} & k^{(2)} & \cdots & k^{(n)} & 0 \end{pmatrix} \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ \vdots \\ h^{(n)} \\ h^{(n+1)} \end{pmatrix},$$

where $x = \left(x^{(i)}\right)_{i=\overline{1,n+1}}, h = \left(h^{(i)}\right)_{i=\overline{1,n+1}}, k = \left(k^{(i)}\right)_{i=\overline{1,n+1}} \in \mathbb{K}^{n+1}$.

Suppose that $\mathbb{K}^{n+1}$ is equipped with the max-norm. Then, from the above formula, it follows that $\left\| F''(x) \right\| = n$, for all $x \in \mathbb{K}^{n+1}$.

Another possible choice for the norming function G is

$$F_{n+1}(x) = \frac{1}{2n}\sum_{i=1}^{n}\left(x^{(i)}\right)^2 - 1, \tag{3.2}$$

in which case we get $\left\| F'' \right\| = 2$.

The Theorem stated in the previous section can be reformulated according to this setting.

Remark. In [15] it is proved that for a given eigenpair (v, λ) the operator $F'(v, \lambda)$ is nonsingular iff λ is simple. Hence our result applies only for such eigenvalues.

4. A NUMERICAL EXAMPLE

Consider the real matrix

$$A = \begin{pmatrix} 1 & 1 & 0.5 \\ 1 & 1 & 0.25 \\ 0.5 & 0.25 & 2 \end{pmatrix}$$

which has the following eigenvalues and eigenvectors

$\lambda_1 = -0.01664728361$, $v_1 = (-0.7212071298, 0.6863492877, 0.09372796349)$,
$\lambda_2 = 1.480121423$, $v_2 = (-0.4442810581, -0.5621094204, 0.6976011330)$, and
$\lambda_3 = 2.536525860$, $v_3 = (0.5314834119, 0.4614733520, 0.7103293096)$.

Taking the initial values $x_0 = (-0.7, 0.6, 0.09, -0.01)$ and applying the studied method for G given by (3.1), we obtain the following results:

k	x_1	x_2	x_3	$x_4 = \lambda$
0	−0.7	0.6	0.09	−0.01
1	−0.7241759075	0.6894343346	0.09406959984	−0.01611417648
2	−0.7212379135	0.6863753479	0.09373253857	−0.01663490068
3	−0.7212071340	0.6863492908	0.09372796420	−0.01664727974
4	−0.7212071298	0.6863492877	0.09372796350	−0.01664728361

For the initial values $x_0 = (-1.7, 1.6, 0.2, -0.01)$ and taking G given by (3.2), we obtain

k	x_1	x_2	x_3	$x_4 = \lambda$
0	−1.7	1.6	0.2	−0.01
1	−1.768398012	1.682966665	0.2298835780	−0.01695674174
2	−1.766594350	1.681210330	0.2295865533	−0.01664900983
3	−1.766589467	1.681205540	0.2295856852	−0.01664728365

REFERENCES

1. M. P. Anselone and B. L. Rall, *The solution of characteristic value-vector problems by Newton method*, Numer. Math. **11** (1968), 38–45.
2. E. Cătinaş and I. Păvăloiu, *On the Chebyshev method for approximating the eigenvalues of linear operators*, Rev. Anal. Numér. Théorie Approximation **25**, *1–2* (1996), 43–56.
3. P. G. Ciarlet, *Introduction à l'analyse numérique matricielle et à l'optimisation*, Mason, Paris-Milan-Barcelone-Mexico, 1990.
4. F. Chatelin, *Valeurs propres de matrices*, Mason, Paris-Milan-Barcelone-Mexico, 1988.
5. L. Collatz, *Functionalanalysis und numerische Mathematik*, Berlin-Göttingen-Heidelberg, Springer Verlag, 1964.

6. A. Diaconu, *On the convergence of an iterative proceeding of Chebyshev type*, Rev. Anal. Numér. Théorie Approximation **24**, *1–2* (1995), 91–102.
7. A. Diaconu and I. Păvăloiu, *Sur quelques méthodes itératives pour la résolution des équations opérationnelles*, Rev. Anal. Numér. Théorie Approximation **1**, *1* (1972), 45–61.
8. V. S. Kartîşov and F. L. Iuhno, *O nekotorîh Modifikaţiah Metoda Niutona dlea Resenia Nelineinoi Spektralnoi Zadaci*, J. Vîcisl. matem. i matem. fiz. 33, *9* (1973), 1403–1409.
9. I. Lazăr, *On a Newton-type method*, Rev. Anal. Numér. Théorie Approximation **23**, *2* (1994), 167–174.
10. I. Păvăloiu, *Observations concerning some approximation methods for the solutions of operator equations*, Rev. Anal. Numér. Théorie Approximation **23**, *2* (1994), 185–196.
11. I. Păvăloiu, *Sur les procédés itératifs à un ordre élevé de convergence*, Mathematica (Cluj) **12 (35)**, *2* (1970), 309–324.
12. A. R. Tapia and L. D. Whitley, *The projected Newton method has order* $1+\sqrt{2}$ *for the symmetric eigenvalue problem*, SIAM J. Numer. Anal. **25**, *6* (1988), 1376–1382.
13. F. J. Traub, *Iterative methods for the solution of equations*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.
14. S. Ul'm, *On the iterative method with simultaneous approximation of the inverse of the operator*, Izv. Acad. Nauk. Estonskoi S.S.R. **16**, *4* (1967), 403–411.
15. T. Yamamoto, *Error bounds for computed eigenvalues and eigenvectors*, Numer. Math. **34** (1980), 189–199.

Received August 15, 1996

"Tiberiu Popoviciu" Institute of Numerical Analysis
P.O. Box 68
3400 Cluj-Napoca 1
Romania