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ON AN APPROXIMATING LINEAR POSITIVE OPERATOR OF CHENEY-SHARMA

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1. INTRODUCTION

It is known that by starting from two combinatorial identities of Abel-Jensen

$$(1.1) \qquad (u+v+m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} u(u+k\beta)^{k-1}\left[v+(m-k)\beta\right]^{m-k},$$

$$(1.2) \qquad (u+v)(u+v+m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u(u+k\beta)^{k-1} v\left[v+(m-k)\beta\right]^{m-1-k},$$

Cheney and Sharma [1] have introduced and investigated two linear polynomial positive operators, of Bernstein type, P_m and Q_m, defined – for any function $f:[0,1] \to \mathbf{R}$ – by the following formulas

$$(1.3) \qquad (P_m f)(x;\beta) = \sum_{k=0}^{m} p_{m,k}(x;\beta) f\left(\frac{k}{m}\right),$$

$$(1.4) \qquad (Q_m f)(x;\beta) = \sum_{k=0}^{m} q_{m,k}(x;\beta) f\left(\frac{k}{m}\right),$$

where

$$(1.5) \qquad p_{m,k}(x;\beta) = \binom{m}{k} \frac{x(x+k\beta)^{k-1}\left[1-x+(m-k)\beta\right]^{m-k}}{(1+m\beta)^m},$$

$$(1.6) \qquad q_{m,k}(x;\beta) = \binom{m}{k} \frac{x(x+k\beta)^{k-1}(1-x)\left[1-x+(m-k)\beta\right]^{m-1-k}}{(1+m\beta)^{m-1}}.$$

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It is obvious that for $\beta = 0$ these operators reduce to the classical operator B_m of Bernstein.

In this paper we prove that the second operator Q_m preserves the linear functions and we establish several expressions for the remainder term in the corresponding approximation formula.

2. THE VALUE OF THE OPERATOR Q_m FOR THE MONOMIAL e_1

In [1] it was pointed out that the operator Q_m preserves only the constant functions, after calculation of some integrals involved. But we shall prove that Q_m preserves the linear functions.

It is easy to see that the following theorem is true.

THEOREM 2.1. *The approximating polynomial $Q_m f$ is interpolatory at both sides of the interval* $[0, 1]$, *for any nonnegative value of the parameter* β.

Proof. In order to prove this result, we have only to observe that we can write

$$(Q_m f)(x;\beta) = \frac{1}{(1+m\beta)^{m-1}}\Big\{(1-x)(1-x+m\beta)^{m-1}f(0) -$$

$$-x(x-1)\sum_{k=1}^{m-1}\binom{m}{k}(x+k\beta)^{k-1}[1-x+(m-k)\beta]^{m-1-k}f\left(\frac{k}{m}\right)+x(x+m\beta)^{m-1}f(1)\Big\}.$$

Let us consider next the monomials $e_j(t) = t^j\,(j \geq 0)$ for any $t \in [0,1]$. We shall now state and prove

THEOREM 2.2. *The operator Q_m reproduces the linear functions.*

Proof. As it has been observed in [1], if we replace in the identity (1.2) $u = x$ and $v = 1 - x$, we find that $Q_m e_0 = e_0$, that is, the operator Q_m reproduces the constants.

We shall prove that we also have $Q_m e_1 = e_1$.

Indeed, one can see that we can write

$$(2.1)\qquad (1+m\beta)^{m-1}(Q_m e_1)(x;\beta) =$$

$$= \sum_{k=1}^{m}\frac{k}{m}\binom{m}{k}x(x+k\beta)^{k-1}(1-x)[1-x+(m-k)\beta]^{m-1-k} =$$

$$= \sum_{k=1}^{m}\binom{m-1}{k-1}x(x+k\beta)^{k-1}(1-x)[1-x+(m-k)\beta]^{m-1-k},$$

because

$$(2.2)\qquad \frac{k}{m}\binom{m}{k} = \binom{m-1}{k-1}.$$

If we change $k - 1 = j$ and then denote again the index of summation by k, we have

$$(2.3)\qquad (1+m\beta)^{m-1}(Q_m e_1)(x;\beta) =$$

$$= \sum_{k=0}^{m-1}\binom{m-1}{k}x(x+\beta+k\beta)^{k}(1-x)[1-x+(m-1-k)\beta]^{m-2-k} =$$

$$= (1+m\beta)\sum_{k=0}^{m-1}\binom{m-1}{k}x(x+\beta+k\beta)^{k-1}(1-x)[1-x+(m-1-k)\beta]^{m-2-k} -$$

$$-\sum_{k=0}^{m-1}\binom{m-1}{k}x(x+\beta+k\beta)^{k-1}(1-x)[1-x+(m-1-k)\beta]^{m-1-k},$$

since

$$x+\beta+k\beta = (1+m\beta) - [1-x+(m-1-k)\beta].$$

In order to find the first sum, we replace in the identity (1.2) m by $m-1$ and $u = x+\beta$, $v = 1-x$.

We get

$$(1+\beta)(1+m\beta)^{m-2} = (x+\beta)\sum_{k=0}^{m-1}\binom{m-1}{k}(x+\beta+k\beta)^{k-1}(1-x)\cdot$$

$$\cdot[1-x+(m-1-k)\beta]^{m-2-k}.$$

If we multiply by x and divide by $x+\beta$, we obtain

$$\sum_{k=0}^{m-1}\binom{m-1}{k}x(x+\beta+k\beta)^{k-1}(1-x)[1-x+(m-1-k)\beta]^{m-2-k} =$$

$$= (1+\beta)(1+m\beta)^{m-2}\frac{x}{x+\beta},$$

which represents the first sum.

For finding the second sum we shall use the identity (1.2). We replace m by $m-1$ and $u = x+\beta$, $v = 1-x$ and we find

$$(1+m\beta)^{m-1} = (x+\beta)\sum_{k=0}^{m-1}\binom{m-1}{k}(x+\beta+k\beta)^{k-1}[1-x+(m-1-k)\beta]^{m-1-k}.$$

It follows that we can write

$$\sum_{k=0}^{m-1}\binom{m-1}{k}x(x+\beta+k\beta)^{k-1}(1-x)\left[1-x+(m-1-k)\beta\right]^{m-1-k}=$$

$$=(1+m\beta)^{m-1}\frac{x(1-x)}{x+\beta}.$$

Consequently, we have

$$(2.4)\quad (Q_m e_1)(x;\beta)=\frac{1}{(1+m\beta)^{m-1}}\left[(1+m\beta)(1+\beta)(1+m\beta)^{m-2}\frac{x}{x+\beta}-\right.$$

$$\left.-(1+m\beta)^{m-1}\frac{x(1-x)}{x+\beta}\right]=\frac{x}{x+\beta}\left[1+\beta-(1-x)\right]=x.$$

Therefore we have

$$(2.5)\quad Q_m e_0=e_0,\ Q_m e_1=e_1,$$

as in the case of the classical Bernstein operator B_m.

3. THE REMAINDER

Since the operator Q_m reproduces the linear functions, it is clear that the approximation formula

$$(3.1)\quad f(x)=(Q_m f)(x;\beta)+(R_m f)(x;\beta)$$

has the degree of exactness $N=1$.

First we shall give an integral representation of the remainder.

THEOREM 3.1. *If the function f has a continuous second derivative on the interval* $[0, 1]$, *then we can represent the remainder of the approximation formula* (3.1) *under the following integral form*

$$(3.2)\quad (R_m f)(x)=\int_0^1 G_m(t;x)f''(t)\,dt,$$

where

$$(3.3)\quad G_m(t;x)=(R_m\varphi_x)(t),\quad \varphi_x(t)=\frac{x-t+|x-t|}{2}=(x-t)_+$$

and R_m operates on $\varphi_x(t)$ as a function of x.

Proof. The representation (3.2) can be obtained at once if we apply the well-known theorem of Peano.

For the Peano kernel associated to the operator Q_m we have

$$(R_m\varphi_x)(t)=(x-t)_+-\sum_{k=0}^{m}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right)_+.$$

In order to find explicit expressions of this kernel, we assume that $x\in\left[\frac{s-1}{m},\frac{s}{m}\right]$ and we can write

$$(3.4)\quad G_m(t;x)=x-t-\sum_{k\ge j}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right)$$

for $t\in\left[\frac{j-1}{m},\frac{j}{m}\right]$, where $1\le j\le s-1$.

If we assume that $t\in\left[\frac{s-1}{m},x\right]$, then we obtain

$$(3.5)\quad G_m(t;x)=x-t-\sum_{k\ge s}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right),$$

while for $t\in\left[x,\frac{s}{m}\right]$ we get

$$(3.6)\quad G_m(t;x)=-\sum_{k\ge s}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right).$$

For $t\in\left[\frac{j-1}{m},\frac{j}{m}\right]$ $(j>s)$ we have

$$(3.7)\quad G_m(t;x)=-\sum_{k\ge j}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right).$$

Because the degree of exactness of formula (3.1) is one, by replacing $f(x)=x-t$, the corresponding remainder vanishes and we obtain

$$x-t=\sum_{k=0}^{m}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right)=\sum_{k=0}^{j-1}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right)+\sum_{k=j}^{m}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right).$$

Therefore we can write

$$x-t=\sum_{k=j}^{m}q_{m,k}(x;\beta)\left(\frac{k}{m}-t\right)=-\sum_{k=0}^{j-1}q_{m,k}(x;\beta)\left(t-\frac{k}{m}\right).$$

Consequently, the representation (3.4) can be replaced by

$$(3.8)\qquad G_m(t;x) = -\sum_{k=0}^{j-1} q_{m,k}(x;\beta)\left(t-\frac{k}{m}\right)$$

if $t \in \left[\frac{j-1}{m}, \frac{j}{m}\right]$ and $1 \le j \le s-1$, while (3.5) can be replaced by

$$(3.9)\qquad G_m(t;x) = -\sum_{k=0}^{s-1} q_{m,k}(x;\beta)\left(t-\frac{k}{m}\right),$$

when $t \in \left[\frac{s-1}{m}, x\right]$.

THEOREM 3.2. *If $f \in C^2[0,1]$, then the remainder of the Cheney-Sharma approximation formula* (3.1) *can be represented under the following form*

$$(3.10)\qquad (R_m f)(x;\beta) = \frac{1}{2}(R_m e_2)(x;\beta) f''(\xi),\quad 0<\xi<1.$$

Proof. From (3.6)–(3.9) it is easy to see that on the square $D = [0,1]\times[0,1]$ the function $y = G_m(t) = G_m(t;x)$ represents a polygonal continuous line situated beneath the x-axis.

By applying the first law of the mean to the integral (3.2), we get

$$(R_m f)(x;\beta) = f''(\xi)\int_0^1 G_m(t;x)\,dt$$

and formula (3.1) becomes

$$(3.11)\qquad f(x) = (Q_m f)(x;\beta) + f''(\xi)\int_0^1 G_m(t;x)\,dt.$$

If we replace here $f(x) = e_2(x) = x^2$, we obtain

$$x^2 = (Q_m e_2)(x;\beta) + 2\int_0^1 (t;x)\,dt.$$

Consequently, we can write

$$(3.12)\qquad \int_0^1 G_m(t;x)\,dt = \frac{1}{2}\left[x^2 - (Q_m e_2)(x)\right] = \frac{1}{2}(R_m e_2)(x;\beta).$$

Formulas (3.11) and (3.12) lead us to the desired approximation formula with the expression (3.10) for the remainder term.

In the special case $\beta = 0$, when $Q_m = B_m$, the formulas corresponding to (3.2)–(3.3) and (3.10) were first established in our old paper [3].

Remark. Since the polynomial $Q_m f$ is interpolatory at both sides of the interval [0, 1], it is clear that $(R_m e_2)(x;\beta)$ contains the factor $x(x-1)$.

Since $R_m e_0 = 0$, $R_m e_1 = 0$ and $R_m f \neq 0$ for any convex function of the first order, we can apply a criterion of T. Popoviciu [2] and we can find that the remainder is of a simple form.

Consequently, we can state

THEOREM 3.3. *If the second-order divided differences of the function f are bounded on the interval* [0, 1], *then there exist three points $t_{m,1}$, $t_{m,2}$ and $t_{m,3}$ from* [0, 1] *which might depend on f, such that the remainder of the approximation formula* (3.1) *can be represented under the form*

$$(3.13)\qquad (R_m f)(x;\beta) = (R_m e_2)(x;\beta)\left[t_{m,1}, t_{m,2}, t_{m,3}; f\right].$$

It is clear that, if $f \in C^2[0,1]$ and we apply the mean-value theorem of divided differences, we can obtain formula (3.10) from formula (3.13).

REFERENCES

1. E. W. Cheney and A. Sharma, *On a generalization of Bernstein polynomials*, Riv. Mat. Univ. Parma 5 (1964), 77–84.
2. T. Popoviciu, *Sur le reste dans certaines formules linéaires d'approximation de l'analyse*, Mathematica 1 (24) (1959), 95–142.
3. D. D. Stancu, *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comput. 17 (1963), 270–278.

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