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DUALITY FOR MULTIPLE RIGHT-HAND CHOICE PSEUDOMONOTONIC PROGRAMMING

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1. INTRODUCTION

The main problem studied in this paper is a pseudomonotonic programming problem, having linear constraints, with a multiple right-hand choice.

The case in which the objective function is a linear function has been studied by E. L. Johnson [6], where it has been called "integer programming with continuous variables".

D. Granot, F. Granot and E. L. Johnson [4] introduced for multiple right-hand choice linear programming a subadditive dual and proved some duality results.

The main purpose of this paper is to generalize these duality results to the case of pseudomonotonic programming problem with a multiple right-hand choice.

2. DEFINITIONS AND PRELIMINARIES

In this section we will briefly summarize some basic definitions and properties of the class of pseudomonotonic functions which are nonlinear and, beyond this, nonconcave.

DEFINITION 1. A function $f: D \to R\left(D \subseteq R^n\right)$ is said to be *pseudoconvex* if

$$(y-x)\nabla f(x) \geq 0 \Rightarrow f(y)-f(x) \geq 0,$$

where ∇f is the gradient vector whose components are the partial derivatives of f and D is a convex subset in R^n.

DEFINITION 2. The function f is said to be *pseudoconcave* if $-f$ is pseudoconvex.

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DEFINITION 3. A function f that is both pseudoconvex and pseudoconcave is called *pseudomonotonic.*

Obviously, linear functions form the most important subclass of these, generally, nonlinear nonconvex (nonconcave) functions. A more general subclass of pseudomonotonic functions is that of the linear fractional functions with a positive nominator on a given convex subset in R^n (cf., e.g., B. Martos [8]).

Other types of pseudomonotonic functions which were not linear-fractional were considered, for instance, by B. Mond [9], Şt. Ţigan ([13], [14], [15]), C. R. Bector and P. L. Jolly [1].

We make the observation that pseudomonotonic functions represent a generalization of convex functions. Another generalization was made by B. D. Craven [2], G. Giorgio and E. Mohlo [3], M. A. Hanson [5] (invex, quasi-invex, and pseudoinvex functions), T. Popoviciu [12] (convex functions of the order n), and E. Popoviciu [11] (quasi-convex functions of the order n).

Now, let S be a nonempty subset of D. In a very general formulation, the pseudomonotonic programming can be stated as

PM. Find

$$\operatorname{Min}\{f(x) \mid x \in S\}.$$

Now we present a useful linearization property of the pseudomonotonic programming PM, obtained by K. O. Kortanek and J. P. Evans [7] in the particular case when the feasible set S is a convex set, and then extended by Şt. Ţigan ([13], [14], [15]) to the nonconvex feasible set.

THEOREM 1. *Let f be a continuously differentiable pseudomonotonic function on the convex set D and let the feasible set S be a closed bounded nonvoid subset of D. Then $x^0 \in S$ is an optimal solution of the problem PM if and only if x^0 is an optimal solution for the following linearized programming problem*

$P(x^0)$. Find

$$\operatorname{Min}\{\nabla f(x^0)x \mid x \in S\}.$$

We mention that Theorem 1 can be used as an optimality criterion in the linearization procedures for solving the pseudomonotonic programming with the nonconvex feasible set (see, e.g., [1], [9], [13], [14], [15]).

3. PSEUDOMONOTONIC PROGRAMMING WITH A MULTIPLE RIGHT-HAND CHOICE

The pseudomonotonic programming problem with a multiple right-hand choice can be stated in the following form

P. Find

$$\operatorname{Min}\{f(x) \mid Ax \geq b, \text{ for some } b \in B, x \geq 0\},$$

where B is a finite set of vectors in R^m, f is a pseudomonotonic function, and A is a given $m \times n$ matrix with real elements.

Let S be the feasible set of problem PM, that is,

$$S = \{x \in R^n \mid Ax \geq b, \text{ for some } b \in B, x \geq 0\}.$$

In general, the feasible set S of problem PM is a nonconvex set in R^n.

DEFINITION 4. The multiple right-hand choice pseudomonotonic problem P is *regular* if for every $b \in B$ the partial feasible set

$$S(b) = \{x \in R^n \mid Ax \geq b, x \geq 0\} \tag{1}$$

is a closed bounded set in R^n, and there is at least one $b \in B$ such that $S(b) \neq \varnothing$.

In [4], for the particular case of the linear programming with a multiple right-hand choice

$$\operatorname{Min}\{cx \mid Ax \geq b, \text{ for some } b \in B, x \geq 0\}, \tag{2}$$

it has been introduced the dual problem of (2) as the subadditive program

$$\operatorname{Max}\left\{\operatorname*{Min}_{b \in B} \pi(b) \,\middle|\, \pi(A^j) \leq c_j,\, j = 1, 2, \ldots, n \text{ and } \pi \in H\right\}, \tag{3}$$

where A^j is the j-th column of A and H is the set of all subadditive, positively homogeneous and monotone functions from R^m to the extended real line, and by convention

$$\lambda + (\pm\infty) = \pm\infty,\ \lambda \cdot (\pm\infty) = \pm\infty,\ \ 0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0,\ -\infty + \infty = +\infty$$

for any real λ.

In the linear case, the following strong duality theorem holds (see [4])

THEOREM 2. *If there is a feasible solution to* (1) *and if there is a lower bound of cx over all feasible solutions to* (1), *then there exists a pair* (x^0, π^0) *of feasible solutions to* (1) *and* (2) *such that* $\operatorname*{Min}_{b \in B} \pi^0(b) = cx^0$.

In the case when $|B| = 1$, Theorem 2 can be strengthened to require π to be linear. However, in the general case for finite B, strong duality fails to hold if π is restricted to be linear (see [4]).

The constraints of problems (2) and P can be considered as a particular case of disjunctive constraints. We mention that V. N. Patkar and I. M. Stancu-Minasian [10] obtained a duality property for fractional programming with disjunctive linear constraints.

4. DUALITY FOR THE PSEUDOMONOTONIC PROGRAMMING WITH A MULTIPLE RIGHT-HAND CHOICE

The dual problem to P is an extension of the Mond dual [9] (see also [13]) for ordinary pseudomonotonic programming. It is introduced as the following pseudomonotonic program with nonlinear constraints and having as dual variables subadditive, positively homogeneous and monotone functions from R^m to the extended real line.

DP. Find

$$\operatorname{Max} f(u)$$

subject to

(4) $$\pi\left(A^j\right) \le \frac{\partial f(u)}{\partial x_j}, \; j = 1, 2, \ldots, n$$

(5) $$u^t \nabla f(u) \le \pi(b), \; \forall b \in B.$$

THEOREM 3 (weak duality). *If f is a pseudomonotonic function on a convex set D including the feasible set S, then for any feasible solution x of P and any feasible solution (u, π) of DP, the inequality $f(x) \ge f(u)$ holds.*

Proof. From $Ax \ge b$ and π monotone we have

(6) $$\pi(Ax) \ge \pi(b) \text{ for some } b \in B.$$

From (4) and $x \ge 0$ it follows that

(7) $$\sum_{j=1}^{n} x_j \pi\left(A^j\right) \le x^t \nabla f(u).$$

But, since π is subadditive and positively homogeneous, it results

(8) $$\pi(Ax) \le \sum_{j=1}^{n} \pi\left(A^j x_j\right) = \sum_{j=1}^{n} x_j \pi\left(A^j\right).$$

The inequalities (6), (7) and (8) imply

$$\pi(b) \le x^t \nabla f(u),$$

from where, by (5), it follows

$$u^t \nabla f(u) \le \pi(b) \le x^t \nabla f(u), \text{ for some } b \in B,$$

which means

(9) $$(x - u)^t \nabla f(u) \ge 0.$$

But, since f is pseudomonotonic, it is pseudoconcave, so that (9) implies the inequality $f(u) \ge f(x)$.

THEOREM 4 (strong duality). *Let P be a regular pseudomonotonic problem with a multiple right-hand choice. If x^0 is an optimal solution for P, then there exists $\pi^0 \in H$ such that (x^0, π^0) is an optimal solution for DP and the optimal values of primal and dual problems are equal.*

Proof. By Theorem 1, x^0 is an optimal solution for P if and only if it is an optimal solution for the multiple choice linear programming problem

$$P\left(x^0\right). \; \operatorname{Min}\left\{\nabla f\left(x^0\right)x \mid Ax \ge b, \text{ for some } b \in B, x \ge 0\right\}.$$

But the linear programming problem with a multiple right-hand choice $P(x^0)$ has the following dual

$$D\left(x^0\right). \; \operatorname{Max}\left\{\operatorname*{Min}_{b \in B} \pi(b) \,\middle|\, \pi\left(A^j\right) \le \frac{\partial f\left(x^0\right)}{\partial x_j}, \; j = 1, 2, \ldots, n, \pi \in H\right\}.$$

On the other hand, since P is regular, the feasible set S is a bounded closed set. Then it follows that there is a lower bound on $\nabla f\left(x^0\right)x$ over S. Therefore, by Theorem 2 (a strong duality theorem for linear programming with a multiple right-hand choice), there exists a pair (x', π^0) of feasible solutions of $P(x^0)$ and $D(x^0)$ such that

$$\operatorname*{Min}_{b \in B} \pi^0(b) = \nabla f\left(x^0\right) \cdot x'.$$

But

$$\nabla f\left(x^0\right)x' \ge \operatorname{Min}\left\{\nabla f\left(x^0\right)x \mid x \in S\right\} = \nabla f\left(x^0\right)x^0.$$

Therefore, (x^0, π^0) is a feasible solution for DP, because

$$\pi^0(b) \ge \nabla f\left(x^0\right)x^0, \quad \forall b \in B.$$

But, by Theorem 3, since x^0 is a feasible solution for P, it follows that

$$f\left(x^0\right) \ge f(u)$$

for any feasible solution (u, π) of DP. Thus it results that (x^0, π^0) is an optimal solution for DP.

We remark that Theorem 4 is considerably weaker than the usual pseudomonotonic strong duality theorem [9].

5. CONCLUSIONS

In this paper we consider a pseudomonotonic problem with a multiple right-hand choice, which represents a particular case of pseudomonotonic programming with disjunctive constraints.

For this problem we introduce a subadditive dual and prove weak and strong duality theorems. A variant of these results can be found in our paper [16].

In order to prove the strong duality theorem, a linearization property for pseudomonotonic programming and a duality theorem for multiple right-hand choice linear programming are used.

Although the results presented here are restricted to the pseudomonotonic programming with classical linear constraints, the methods can be adapted and applied to other classes of pseudomonotonic programming.

In particular, it is possible to derive similar duality results for the multiple right-hand choice pseudomonotonic programming with generalized linear constraints in Dantzig's sense or symmetric pseudomonotonic programming [14]. Some of these possibilities will be explored in subsequent papers.

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