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ON THE BOHR-MOLLERUP-ARTIN CHARACTERIZATION OF THE GAMMA FUNCTION

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1. INTRODUCTION

Bohr and Mollerup [2] were the first to prove that the only log-convex solution $f:\mathbf{R}^+ \to \mathbf{R}^+$ to the functional equation $f(x+1) = x f(x)$ for $x > 0$ satisfying $f(1) = 1$ is the gamma function Γ. Their proof was simplified by Artin [1], who based his celebrated treatment of the gamma function on the result. Consequently, this has become known as the Bohr-Mollerup-Artin Theorem, and was adopted by Bourbaki [3] as the starting point for his exposition of the gamma function. Further discussion of the theorem can be found in Leipnik and Oberg [6], and Webster [9], while an appreciation of the result and a historical profile of the gamma function are to be found in Davis [4].

A question that naturally arises is: *for which functions* $g:\mathbf{R}^+ \to \mathbf{R}^+$ *is there a unique eventually log-convex solution* $g^*:\mathbf{R}^+ \to \mathbf{R}^+$ *to the functional equation* $g^*(x+1) = g(x)g^*(x)$ *for* $x > 0$ *satisfying* $g^*(1) = 1$? Our main result, a generalization of the Bohr-Mollerup-Artin Theorem, is that a sufficient condition on g for this to occur is that it is eventually log-concave and has the property that for each $w > 0$, $g(x+w)/g(x) \to 1$ as $x \to \infty$. Moreover, g^* is determined by the formula

$$g^*(x) = \lim_{n\to\infty} \frac{g(n)\dots g(1)g^x(n)}{g(n+x)\dots g(x)} \quad \text{for } x > 0. \tag{1.1}$$

We call a function g^* arising in this way a Γ-*type function*. When g is the identity function on $\mathbf{R}^+$, g^* is simply the gamma function Γ restricted to $\mathbf{R}^+$, this showing that Γ is itself a Γ-type function; in this case (1.1) becomes Gauss well-known limit for $\Gamma(x)$.

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One can establish for Γ-type functions analogues of: *Euler's constant, Weierstrass' infinite product, Wallis's Formula, Gauss' and Legendre's Multiplication Formulas*, and *Stirling's Formula* for gamma function. The backdrop of Γ-type functions serves to place the classical theory into perspective with individual results often appearing more intuitive in this wider setting. For example, it makes clear why some analogue of *Legendre's Duplication Formula* must hold, almost rendering formal proof unnecessary! Not surprisingly, Γ-type functions play a part in finding log-convex solutions to particular functional equations of the form $f(x+1) = g(x)f(x)$ such as those studied by John [5] and Mayer [7].

In this introductory paper, we content ourselves with proving the existence of Γ-type functions and establishing for them an analogue of the classical *Gauss' Multiplication Formula.*

2. CONVEX AND LOG-CONVEX FUNCTIONS

Here we collect together those results about convex and log-convex functions that will be helpful in our discussion of Γ-type functions. In addition to recalling some of their well-known properties, we also mention a few seemingly unrecorded ones. An excellent account of convex functions is given in Roberts and Varberg [8].

Throughout, I denotes an interval on the real line $\mathbf{R}$ whose interior is nonempty. A function $f: I \to \mathbf{R}$ is *convex* if $f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y)$ whenever $x, y \in I$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$, and is *eventually convex* if f contains a subinterval that is unbounded above, and on which the restriction of f is convex.

An easy consequence of the definition of convexity is that if $f: I \to \mathbf{R}$ is convex, I is unbounded above and $\delta > 0$, then the function $f: I \to \mathbf{R}$ defined by the equation

$$f_\delta(x) = f(x+\delta) - f(x) \quad \text{for} \quad x \in I$$

is increasing. A convex function is continuous on the interior of its domain. A differentiable function $f: I \to \mathbf{R}$ is convex if and only if f is increasing on I, so a twice differentiable function $f: I \to \mathbf{R}$ is convex if and only if f is nonnegative on I. A function $f: I \to \mathbf{R}$ is said to be *(eventually) concave* if its negative $-f: I \to \mathbf{R}$ is (eventually) convex. The results stated above for convex functions have obvious analogues for concave functions, and these we take as read.

Central to our discussion are the concepts of log-convexity and log-concavity. Whereas the notion of log-convexity does appear fleetingly in the literature on convex functions, we have been unable to locate a single *explicit* reference to the idea of log-concavity or even a single occurrence of the word *log-concave*.

A function $f: I \to \mathbf{R}^+$, where $\mathbf{R}^+$ denotes the set of positive numbers, is called *log-convex (log-concave)* if $\log f: I \to \mathbf{R}$ is convex (concave) and *eventually log-convex (eventually log-concave)* if $\log f: I \to \mathbf{R}$ is eventually convex (eventually concave). Thus f is (eventually) log-convex if and only if f^{-1} is (eventually) log-concave. The class of all log-convex functions defined on a given interval I is closed under both addition and multiplication, whereas the class of all log-concave functions defined on I is only closed under multiplication. A differentiable function $f: I \to \mathbf{R}^+$ is log-convex (log-concave) if and only if the function f'/f is increasing (decreasing) on I, so a twice differentiable function $f: I \to \mathbf{R}$ is log-convex (log-concave) if and only if $f''f - (f')^2$ is nonnegative (nonpositive) on I.

The definitions of log-convexity and log-concavity given earlier make it clear why these terms are chosen, but do not provide the formulations of these concepts most useful in practice – these we obtain below. Clearly, a function $f: I \to \mathbf{R}$ is log-convex if and only if

$$f(\lambda x + \mu y) \le f^\lambda(x) f^\mu(y) \tag{2.1}$$

whenever $x, y \in I$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$. The *arithmetic mean-geometric mean inequality* shows that, for $f(x), f(y), \lambda, \mu > 0$ with $\lambda + \mu = 1$

$$f^\lambda(x) f^\mu(y) \le \lambda f(x) + \mu f(y),$$

whence every log-convex function is convex. The function $f: \mathbf{R} \to \mathbf{R}$ defined by the equation $f(x) = |x| + 1$ for $x \in \mathbf{R}$ is positive and convex on any interval I, but is *not* log-convex there. Thus the concept of log-convexity is *stronger* than that of convexity. The reformulation of log-convexity implied by inequality (2.1) is equivalent to the following working definition: *the function* $f: I \to \mathbf{R}$ *is log-convex if and only if for all* $x, y, z \in I$ *with* $x < y < z$

$$f^{z-x}(y) \le f^{z-y}(x) f^{y-x}(z). \tag{2.2}$$

The definition of log-concavity can be reformulated in an analogous way, with the $\le$ sign in (2.2) being replaced by a $\ge$ sign. It must be emphasized that the concept of log-concavity is *weaker* than that of concavity. For example, the exponential function is log-concave on any interval I, but is not concave there. We now introduce a subclass $\mathcal{G}$ of the class of all eventually log-concave function on $\mathbf{R}^+$ which plays the central role in constructing Γ-type functions. To be precise, $\mathcal{G}$ consists of all eventually log-concave functions $g: \mathbf{R}^+ \to \mathbf{R}^+$ with the property that, for each $w > 0$, $g(x+w)/g(x) \to 1$ as $x \to \infty$. The single most important function in $\mathcal{G}$ is the *function* x, here the identity function on $\mathbf{R}^+$, which gives rise to the Γ-type function itself. Other functions in $\mathscr{G}$ are the restrictions to $\mathbf{R}^+$ of $x^2 + a^2$, $x/(x+a)$ and tanh ax where $a > 0$. More generally, the restriction to $\mathbf{R}^+$ of any real polynomial that is positive on $\mathbf{R}^+$ lies in $\mathscr{G}$. Also, if f is a real rational function that is positive on $\mathbf{R}^+$, then the restriction to $\mathbf{R}^+$ of either f or $1/f$ lies in $\mathscr{G}$. Clearly, the class $\mathscr{G}$ is closed under multiplication. One crucial property of members of $\mathscr{G}$ is that they are eventually increasing as the next theorem shows.

THEOREM 2.1. *Let $g \in \mathscr{G}$ be log-concave on some unbounded subinterval I of $\mathbf{R}^+$. Then g is increasing on I.*

Proof. Let $a, b \in I$ with $a < b$. Since log g is concave on I, for all $x > 0$,

$$\log g(b) - \log g(a) \geq \log g(b+x) - \log g(a+x),$$

so

$$g(b)/g(a) \geq g(b+x)/g(a+x).$$

Since $g \in \mathscr{G}$, the right-hand side of the last inequality tends to 1 as $x \to \infty$, whence $g(b) \geq g(a)$ and g is increasing on I. □

3. Γ-TYPE FUNCTIONS

Theorem 3.1 below generalizes the Bohr-Mollerup-Artin Theorem in that it concerns a whole class of functional equations of the form $f(x+1) = g(x)f(x)$ for $x > 0$, not just the specific one $f(x+1) = xf(x)$. Incidentally, it assumes only that the solutions to the equation are *eventually* log-convex, not log-convex on $\mathbf{R}^+$ in the classical result.

THEOREM 3.1. *Let the function $g: \mathbf{R}^+ \to \mathbf{R}^+$ have the property that, for each $w > 0$, $g(x+w)/g(x) \to 1$ as $x \to \infty$. Suppose that $f: \mathbf{R}^+ \to \mathbf{R}^+$ is an eventually log-convex function satisfying the functional equation $f(x+1) = g(x)f(x)$ for $x > 0$ and the initial condition $f(1) = 1$. Then f is uniquely determined by g through the equation*

$$f(x) = \lim_{n\to\infty} \frac{g(n)\ldots g(1)g^x(n)}{g(n+x)\ldots g(x)} \quad \textit{for } x > 0.$$

Proof. Let $x > 0$. Denote by m the largest integer not exceeding x, and let n be a natural number for which f is log-convex on $[n+m, \infty)$. The log-convexity of f on this interval shows that

$$f(n+x+1) \leq f^{m+1-x}(n+m+1)f^{x-n}(n+m+2)$$

and

$$f(n+m+1) \leq f^{x-m}(n+x)f^{m+1-x}(n+x+1).$$

Since $f(x+1) = g(x)f(x)$ for $x > 0$ and $f(1) = 1$,

$$f(n+m+1) = g(n+m)\ldots g(1) \text{ and } f(n+x+1) = g(n+x)\ldots g(x)f(x);$$

also

$$f^{m+1-x}(n+m+1)f^{x-m}(n+m+2) = f(n+m+1)g^{x-n}(n+m+1)$$

and

$$f^{x-m}(n+x)f^{m+1-x}(n+x+1) = f(n+x+1)g^{n-x}(n+x).$$

It follows that

$$\frac{g(n+m)\ldots g(n+1)}{g(n+x)\ldots g(n+x)}\left(\frac{g(n+x)}{g(n)}\right)^x \leq \frac{g(n+x)\ldots g(x)f(x)}{g(n)\ldots g(1)g^x(n)} \leq$$

$$\leq \frac{g(n+m)\ldots g(n+1)}{g(n+m+1)\ldots g(n+m+1)}\left(\frac{g(n+m+1)}{g(n)}\right)^x.$$

The property assumed of g ensures that both ends of the last string of inequalities converge to 1 as n tends to infinity, whence

$$f(x) = \lim_{n\to\infty} \frac{g(n)\ldots g(1)g^x(n)}{g(n+x)\ldots g(x)}. \; □$$

The property assumed of $g: \mathbf{R}^+ \to \mathbf{R}^+$ in Theorem 3.1 ensures that it is *at most* one eventually log-convex solution $f: \mathbf{R}^+ \to \mathbf{R}^+$ to the functional equation $f(x+1) = g(x)f(x)$ for $x > 0$ satisfying $f(1) = 1$, but does not itself guarantee that such a solution exists. If, however, g is also assumed to be *eventually log-concave*, so g in fact belongs to the class of functions $\mathscr{G}$ introduced in Section 2, then the existence of such a solution is assured by the following result:

THEOREM 3.2. *Let $g \in \mathscr{G}$. Then there exists a unique eventually log-convex function $f: \mathbf{R}^+ \to \mathbf{R}^+$ satisfying the functional equation $f(x+1) = g(x)f(x)$ for $x > 0$ and the initial condition $f(1) = 1$. Moreover,*

$$f(x) = \lim_{n\to\infty} \frac{g(n)\ldots g(1)g^x(n)}{g(n+x)\ldots g(x)} \quad \textit{for } x > 0$$

and f is log-convex on any unbounded subinterval of $\mathbf{R}^+$ on which g is log-concave.

Proof. We establish the existence of f, its uniqueness following from Theorem 3.1. For each $n \in \mathbf{N}$, define a function $f_n : \mathbf{R}^+ \to \mathbf{R}^+$ by the equation

$$f_n(x) = \frac{g(n)\dots g(1)g^x(n)}{g(n+x)\dots g(x)} \quad \text{for} \quad x > 0. \tag{3.1}$$

Then, for $n \in \mathbf{N}$, and $x > 0$

$$f_{n+1}(x) = \frac{g^{x+1}(n+1)}{g(n+x+1)g^x(n)} f_n(x) \tag{3.2}$$

and

$$f_n(x+1) = \frac{g(n)}{g(n+x+1)} g(x) f_n(x). \tag{3.3}$$

Since g is eventually log-concave, there is some nonnegative integer m such that g is log-concave on the interval (m, ∞).

First, let $0 < x \le 1$. We show that the sequence $f_1(x), f_2(x), \dots$ is eventually increasing, and bounded above. Let $n \in \mathbf{N}$ satisfy the inequality $n \ge m + 1$. Then the log-concavity of g on (m, ∞) shows that

$$g^{x+1}(n+1) \ge g^x(n) g(n+x+1),$$

whence (3.2) shows that $f_{n+1}(x) \ge f_n(x)$. Thus the sequence $f_1(x), f_2(x), \dots$ is eventually increasing. The log-concavity of g also yields the $n - m$ inequalities

$$g(m+x+1) \ge g^{1-x}(m+1) g^x(m+2)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$g(n+x) \ge g^{1-x}(n) g^x(n+1),$$

and their product, together with (3.1), shows that

$$f_n(x) \le \frac{g(1)\dots g(m+1)}{g(x)\dots g(m+x)} \left(\frac{g(n)}{g(n+1)} \right)^x g^{x-1}(n+1).$$

But g is increasing on (m, ∞), whence

$$f_n(x) \le \frac{g(1)\dots g(m+1)}{g(x)\dots g(m+x)} g^{x-1}(n+1),$$

which shows that the sequence $f_1(x), f_2(x), \dots$ is bounded above. Thus, for each x in $(0, 1]$, the sequence $f_1(x), f_2(x), \dots$ converges and a function f: $(0, 1]$ may be defined by the equation

$$f(x) = \lim_{n\to\infty} \frac{g(n)\dots g(1)g^x(n)}{g(n+x)\dots g(x)} \quad \text{for} \quad 0 < x \le 1.$$

It follows easily from this last result and (3.3) that a function $f : \mathbf{R}^+ \to \mathbf{R}^+$ may be defined by the equation

$$f(x) = \lim_{n\to\infty} \frac{g(n)\dots g(1)g^x(n)}{g(n+x)\dots g(x)} \quad \text{for} \quad x > 0$$

and that $f(x+1) = g(x)f(x)$ for $x > 0$. Also

$$f(1) = \lim_{n\to\infty} \frac{g(n)\dots g(1)g(n)}{g(n+1)\dots g(1)} = \lim_{n\to\infty} \frac{g(n)}{g(n+1)} = 1.$$

Finally, suppose that g is log-concave on some unbounded interval I. Then every f_n, being a product of functions, each log-convex on I, is itself log-convex on I. Since f is a pointwise limit of functions log-convex on I, it too is log-convex on I. □

We are now in a position to introduce the main object of our study here, Γ*-type functions*. Theorem 3.2 shows that each member g of $\mathscr{G}$ gives rise to a *unique* eventually log-convex function $f : \mathbf{R}^+ \to \mathbf{R}^+$ satisfying the functional equation $f(x+1) = g(x)f(x)$ for $x > 0$ and the initial condition $f(1) = 1$. We indicate the dependence of f upon g by writing $f = g$. Equivalently, g^* can be defined explicitly by the equation

$$g^*(x) = \lim_{n\to\infty} \frac{g(n)\dots g(1)g^x(n)}{g(n+x)\dots g(x)} \quad \text{for} \quad x > 0.$$

We call functions g of the form just described Γ*-type functions*. Clearly, a function $f : \mathbf{R}^+ \to \mathbf{R}^+$ is of Γ-type, precisely when it satisfies the following *four* conditions:

a) $f(1) = 1$;
b) f is eventually log-convex;
c) $f(x+1)/f(x)$ is eventually log-concave;
d) for each $w > 0$, $f(x+w+1)f(x)/f(x+w)f(x+1) \to 1$ as $x \to \infty$.

Examples of Γ-type functions are readily found. If $g : \mathbf{R}^+ \to \mathbf{R}^+$ is the *identity function* on $\mathbf{R}^+$, i.e., $g(x) = x$ for $x > 0$, then $g \in \mathscr{G}$ and $g^* = \Gamma$, showing that Γ is itself a Γ-type function. If $g : \mathbf{R}^+ \to \mathbf{R}^+$ is constantly equal to a positive number c, then $g \in \mathscr{G}$ and $g^*(x) = c^{x-1}$ for $x > 0$.

Before constructing further Γ-type functions, it will be helpful to consider a few of their most basic properties.

THEOREM 3.3. *Let $g_1, g_2, \dots g_n, g \in \mathcal{G}$ and let $g > 0$. Let $h:\mathbf{R}^+ \to \mathbf{R}^+$ be the function defined by the equation $h(x) = g(x + a)$ for $x > 0$. Then:*

(i) $g_1 \dots g_n \in \mathcal{G}$ *and* $(g_1 \dots g_n)^* = g_1^* \dots g_n^*$;

(ii) *if* $g_1 / g_2 \in \mathcal{G}$, *then* $(g_1 / g_2)^* = g_1^* / g_2^*$;

(iii) $h \in \mathcal{G}$ *and* $h(x) = g^*(x + a) / g(a)g^*(a)$ *for* $x > 0$.

Proof. We indicate a general method of establishing identities such as occur in (i)–(iii), by proving the second part of (iii). Define a function $j:\mathbf{R}^+ \to \mathbf{R}^+$ by the equation $j(x) = g^*(x + a) / g(a)g^*(a)$ for $x > 0$. Then j is eventually log-convex $j(x + 1) = h(x)j(x)$ for $x > 0$ and $j(1) = 1$. By Theorem 3.2 $h^*(x) = j(x)$, as required. □

Consider next the function $g:\mathbf{R}^+ \to \mathbf{R}^+$ defined by the equation $g(x) = x/(x+a)$ for $x > 0$, where a is a positive number. Then $g \in \mathcal{G}$ and Theorem 3.3 (ii), (iii) show that $g^*(x) = a\Gamma(a)\Gamma(x) / \Gamma(x + a)$ for $x > 0$. In particular, the *function* $a\Gamma(a)\Gamma(x) / \Gamma(x + a)$ is log-convex, a not completely trivial result, but a fact immediate from the approach taken here.

Now for a more substantial example of a Γ-type function: Let $a > 0$. Define a function $g:\mathbf{R}^+ \to \mathbf{R}^+$ by the equation $g(x) = x^2 + a^2$ for $x > 0$. Then g is log-concave on $[a, \infty)$, but on no interval *strictly* containing this one. Thus, $g \in \mathcal{G}$ and $g^*:\mathbf{R}^+ \to \mathbf{R}^+$ is log-convex on $[a,\infty)$. Also, for $x > 0$,

$$g^*(x) = \lim_{n\to\infty} \frac{(n^2 + a^2)\dots(1^2 + a^2)(n^2 + a^2)^x}{((n + x)^2 + a^2)\dots(x^2 + a^2)} = \frac{\sin h\pi a}{\pi a}\Gamma^2(x)\prod_{n=0}^{\infty}\frac{(n + x)^2}{(n + x)^2 + a^2}.$$

Here we have used Gauss' product for the gamma function and the sine product

$$\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right)$$

with $z = ai$. The second derivative of $\log g^*$ at $x > 0$ is

$$2\sum_{n=0}^{\infty}\frac{(n + x)^2 - a^2}{((n + x)^2 + a^2)^2},$$

which is positive at $x = a$, and so, by continuity, is positive to the *immediate left* of a. It follows that the largest unbounded interval on which g^* is log-convex strictly contains the largest unbounded interval on which g is log-concave.

4. GAUSS' MULTIPLICATION FORMULA

The Bohr-Mollerup-Artin Theorem enables several classical results about the gamma function to be established with ease, none more so than the *Gauss' Multiplication Formula*, and its special case, *Legendre's Duplication Formula.* A similar situation pertains in the context of Γ-type functions, where the analogue results are almost self-suggestive, and proofs are hardly needed. In this more general setting, however, Gauss' formula appears in a slightly modified form, the more usual one applying only to a special class of Γ-type functions.

THEOREM 4.1 (Gauss' Multiplication Formula). *Let $g \in \mathcal{G}$ and $m \in$* N. *Define functions $g_m, h_m:\mathbf{R}^+ \to \mathbf{R}^+$ by the equations*

$$g_m(x) = g\left(\frac{x}{m}\right) \quad and \quad h_m(x) = \frac{g(x)}{g_m(x)} \quad for \quad x > 0.$$

Then $g_m \in \mathcal{G}$, and for $x > 0$

$$g^*\left(\frac{x}{m}\right)g^*\left(\frac{x+1}{m}\right)\dots g^*\left(\frac{g+m-1}{m}\right) = g^*\left(\frac{1}{m}\right)g^*\left(\frac{2}{m}\right)\dots g^*\left(\frac{m-1}{m}\right)g_m^*(x).$$

If $h_m \in \mathcal{G}$, then for $x > 0$

$$g^*\left(\frac{x}{m}\right)g^*\left(\frac{x+1}{m}\right)\dots g^*\left(\frac{x+m-1}{m}\right)h_m^*(x) = g^*\left(\frac{1}{m}\right)g^*\left(\frac{2}{m}\right)\dots g^*\left(\frac{m-1}{m}\right)g^*(x).$$

Proof. Clearly, $g_m \in \mathcal{G}$. Determine a function $f:\mathbf{R}^+ \to \mathbf{R}^+$ by the equation

$$g^*\left(\frac{1}{m}\right)g^*\left(\frac{2}{m}\right)\dots g^*\left(\frac{m-1}{m}\right)f(x) = g^*\left(\frac{x}{m}\right)g^*\left(\frac{x+1}{m}\right)\dots g^*\left(\frac{x+m-1}{m}\right),$$

where $x > 0$. Then f is log-convex $f(x + 1) = g_m(x)f(x)$ for $x > 0$ and $f(1) = 1$, whence, by Theorem 3.1, $f(x) = g_m^*(x)$ for $x > 0$, and the first form of the multiplication formula is established. If $h_m \in \mathcal{G}$, then $h_m^*(x)g_m^*(x) = g^*(x)$ for $x > 0$, which immediately leads to the second form of the formula. □

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