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# APPROXIMATION OF COMPLEX VARIABLE FUNCTIONS AND APPLICATIONS FOR SOLVING SINGULAR INTEGRAL EQUATIONS

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The direct methods of solving singular integral equations (SIE) use results concerning the approximation of functions of complex variables by polynomials. These functions are defined on closed smooth contours. Some results have been given in our papers [1–3] without proof. In this Note we will prove some assertions about the approximation by polynomials of functions defined on arbitrary closed smooth contours. These results have been used in numerical analysis (collocation, quadrature, reduction and spline methods) for solving SIE.

**1.** Further we need the following definitions and notations: By $\Gamma$ we will denote an arbitrary closed smooth contour bounding a simply-connected region $F_+$ containing the point $t = 0$. By $F_-$ we will denote the complement of $F_+ \cup \Gamma$ in the entire complex plane. Let $C(\Gamma)$ be the space of continuous functions on $\Gamma$. Assume that $H_\beta(\Gamma)$ is a set of functions defined on $\Gamma$ and satisfying a Hölder condition with the exponent $\beta$, $0 < \beta \leq 1$. The norm on $H_\beta(\Gamma)$ is defined [4, p. 173] by

$$(1)\qquad \|\varphi\|_\beta = \max_{t\in\Gamma}|\varphi(t)| + \sup_{\substack{t_1 \neq t_2\\ t_1, t_2 \in \Gamma}} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\beta} = \|\varphi\|_C + H(\varphi;\beta).$$

From [4, p. 173] we know that $H_\beta(\Gamma)$ with the norm (1) is a Banach space. According to [5, p. 109], if $0 < \beta < \alpha \leq 1$, then $H_\alpha(\Gamma) \subset H_\beta(\Gamma)$. In this case it also holds $\|\varphi\|_\beta < \mu_1\|\varphi\|_\alpha$, where $\varphi \in H_\alpha(\Gamma)$. This is a consequence of the relation

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$$\sup_{\substack{t_1 \neq t_2 \\ t_1, t_2 \in \Gamma}} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\beta} \leq \max\left\{ \sup_{|t_1 - t_2| \geq 1} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\beta}, \right.$$

$$\left. \sup_{0 < |t_1 - t_2| < 1} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\alpha} \right\} \leq \max\left\{ 2 \max_{t \in \Gamma} |\varphi(t)|, \sup_{\substack{t_1 \neq t_2 \\ t_1, t_2 \in \Gamma}} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\alpha} \right\}.$$

Let $X_n = X_n(\Gamma)$ be the set of all polynomials of the type

(2) $$\varphi_n(t) = \sum_{k=-n}^{n} r_k t^k, \; t \in \Gamma,$$

where $r_k$ are arbitrary real or complex numbers.

For the function $\varphi$ the value $E_n(\varphi) = \inf_{\varphi_n \in X_n} \|\varphi - \varphi_n\|_C$ is called the best uniform approximation by polynomials of the form (2). The polynomial $\varphi_n^*$ for which $E_n(\varphi) = \|\varphi - \varphi_n^*\|_C$ is called the best polynomial uniform approximation of $\varphi$. Recall [6, p. 432] that for each function $\varphi \in C(\Gamma)$ in $X_n$ there always exists such a unique polynomial $\varphi_n^*$.

The module of continuity [5, p. 107] of function $\varphi$ is

$$\omega(\delta, \varphi) = \sup_{|t_1 - t_2| < \delta} |\varphi(t_1) - \varphi(t_2)| \quad (\delta > 0;\ t_1, t_2 \in \Gamma).$$

If $\varphi \in H_\alpha(\Gamma)$ $(0 < \alpha \leq 1)$, then

(3) $$\omega(\delta, \varphi) \leq H(\varphi; \alpha)\delta^\alpha.$$

It is known [7, p. 311] that if $\varphi \in C(\Gamma)$, then $E_n(\varphi) \leq \mu_2 \cdot \omega\left(\frac{1}{n}, \varphi\right)$, where $\mu_2$ are constants not depending on $n$ and on the function $\varphi$. From these two last inequalities we deduce that if $\varphi \in H_\alpha(\Gamma)$, then

(4) $$E_n(\varphi) \leq \mu_2 \frac{1}{n^\alpha} H(\varphi; \alpha).$$

THEOREM 1. *Let $\varphi \in H_\alpha(\Gamma)$ and $\varphi_n^*$ be the polynomial of the best uniform approximation of $\varphi$. Then*

(5) $$\|\varphi - \varphi_n^*\|_\beta \leq \mu_3 \frac{1}{n^{\alpha-\beta}} H(\varphi; \alpha) \quad (0 < \beta \leq \alpha < 1).$$



*Proof.* Let us first assume that $|t_1 - t_2| \geq \frac{1}{n}(t_1, t_2 \in \Gamma)$. Then, from (4), we have

$$\frac{|\varphi(t_1) - \varphi_n^*(t_1) - \varphi(t_2) + \varphi_n^*(t_2)|}{|t_1 - t_2|^\beta} \leq 2 \max_{t \in \Gamma} |\varphi(t) - \varphi_n^*(t)| n^\beta \leq \frac{2\mu_2}{n^{\alpha-\beta}} H(\varphi; \alpha).$$

If $|t_1 - t_2| < \frac{1}{n}$, according to [3, p. 42], we receive

(6) $$\omega(\delta, \varphi_n^*) \leq \mu_4 \cdot \omega(\delta, \varphi).$$

Since $\varphi \in H_\alpha(\Gamma)$, then $|\varphi(t_1) - \varphi(t_2)| \leq |t_1 - t_2|^\alpha H(\varphi; \alpha)$. From (3) and (6) we deduce $|\varphi_n^*(t_1) - \varphi_n^*(t_2)| \leq \mu_4 |t_1 - t_2|^\alpha H(\varphi; \alpha)$. Therefore,

$$\frac{|\varphi(t_1) - \varphi_n^*(t_1) - \varphi(t_2) + \varphi_n^*(t_2)|}{|t_1 - t_2|^\beta} \leq \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^\beta} + \frac{|\varphi_n^*(t_1) - \varphi_n^*(t_2)|}{|t_1 - t_2|^\beta} \leq$$

$$\leq \frac{|t_1 - t_2|^\alpha}{|t_1 - t_2|^\beta} H(\varphi; \alpha) + \frac{\mu_4 |t_1 - t_2|^\alpha}{|t_1 - t_2|^\beta} H(\varphi; \alpha) < \mu_5 \frac{1}{n^{\alpha-\beta}} H(\varphi; \alpha).$$

LEMMA 1. *For each polynomial $\varphi_n$ from $X_n$ we have*

$$\|\varphi_n'\|_C \leq \mu_6 \cdot n \|\varphi_n\|_C.$$

*Proof*

$$\varphi_n(t) = \sum_{k=-n}^{n} r_k t^k = t^{-n}\left(r_{-n} + r_{-n+1}t + \ldots + r_n t^{2n}\right) \equiv t^{-n} P_{2n}(t),$$

$$\varphi'_n(t) = -nt^{-n-1} P_{2n}(t) + t^{-n}(P_{2n}(t))' = -n \frac{1}{t} \varphi_n(t) + t^{-n}(P_{2n}(t))'.$$

Assume that the function $t = \psi(w) = cw + c_0 + c_1 w^{-1} + \ldots$ performs a conformal mapping of the exterior of the unit circle $\{|w| > 1\}$ onto $F_-$ so that $\psi(\infty) = \infty$ and $\psi'(\infty) = c > 0$.

According to the Cauchy theorem about derivatives of the analytic functions, we have

(7) $$P'_{2n}(t) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{P_{2n}(\tau)}{(\tau - t)^2} d\tau = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\varphi_n(\tau)}{\tau^{-n}(\tau - t)^2} d\tau,$$

where $\Gamma_\rho = \{t; t = \psi(\rho w), |w| = 1, \rho > 1\}$ is the level line for some $\rho > 1$. We may construct a circle $|\tau - t| = b(\rho - 1), b > 0$, with the center on an arbitrary point $t \in \Gamma$ such that it does not intersect the level line $\Gamma_\rho$. Let us choose $t = \psi(w), \tau = \psi(\rho w)$. Then we obtain

$$|\tau - t| = |\psi(\rho w) - \psi(w)| = |\psi'(\xi)||\rho w - w| = |\psi'(\xi)|(\rho - 1).$$

From [7, p. 181] $0 < m \le |\psi'(\xi)| \le M < \infty, |\xi| \ge 1$. Then

$$m(\rho - 1) \le |\tau - t| \le M(\rho - 1).$$

In this case, the circle $|\tau - t| = \frac{m}{2}(\rho - 1)$ does not intersect $\Gamma_\rho$. By the Cauchy theorem and from (7) it follows that

$$P'_{2n}(t) = \frac{1}{2\pi i}\int_{|\tau - t| = b(\rho - 1)} \frac{\varphi_n(\tau)}{\tau^{-n}(\tau - t)^2}\, d\tau.$$

Further,

$$|t^{-n}P'_{2n}(t)| = |t^{-n}|\frac{1}{2\pi}\int_{|\tau - t| = b(\rho - 1)} \frac{|\varphi_n(\tau)|}{|\tau^{-n}||\tau - t|^2}|d\tau| \le$$

$$\le \max_{\tau \in \{|\tau - t| = b(\rho - 1)\}} |\varphi_n(\tau)| \frac{1}{2\pi}\int_{|\tau - t| = b(\rho - 1)} \left|\frac{\tau^n}{t^n}\right| \frac{|d\tau|}{|\tau - t|^2} = B\frac{1}{2\pi}\frac{1}{[b(\rho - 1)]^2}\int_{|\tau - t| = b(\rho - 1)} \left|\frac{\tau}{t}\right|^n |d\tau|,$$

where

$$B = \max_{\tau \in \{|\tau - t| = b(\rho - 1)\}} |\varphi_n(\tau)|.$$

Let $\tau - t = b(\rho - 1)e^{i\varphi}\ (0 \le \varphi \le 2\pi)$. Then $\tau = t + b(\rho - 1)e^{i\varphi}$; $d\tau = b(\rho - 1)i\, e^{i\varphi}d\varphi$; $|d\tau| = (\rho - 1)b\, d\varphi$. Let $t = x + iy$. Then

$$|t^{-n}P'_{2n}(t)| \le B\frac{1}{2\pi}\frac{1}{[b(\rho - 1)]^2}\int_0^{2\pi}\left|\frac{t + b(\rho - 1)e^{i\varphi}}{t}\right|^n b(\rho - 1)\, d\varphi =$$

$$= B\frac{1}{2\pi}\frac{1}{b(\rho - 1)}\int_0^{2\pi}\left|\frac{x + iy + b(\rho - 1)(\cos\varphi + i\sin\varphi)}{x + iy}\right|^n d\varphi =$$



$$= B\frac{1}{2\pi}\frac{1}{b(\rho - 1)}\int_0^{2\pi}\left[\frac{(x + b(\rho - 1)\cos\varphi)^2 + (y + b(\rho - 1)\sin\varphi)^2}{x^2 + y^2}\right]^{n/2} d\varphi =$$

$$= B\frac{1}{2\pi}\frac{1}{b(\rho - 1)}\int_0^{2\pi}\left[\frac{x^2 + y^2 + b^2(\rho - 1)^2 + 2b(\rho - 1)(x\cos\varphi + y\sin\varphi)}{x^2 + y^2}\right]^{n/2} d\varphi =$$

$$= B\frac{1}{2\pi}\frac{1}{b(\rho - 1)}\int_0^{2\pi}\left[1 + 2b(\rho - 1)\frac{\frac{b(\rho - 1)}{2} + (x\cos\varphi + y\sin\varphi)}{x^2 + y^2}\right]^{n/2} d\varphi.$$

Let $\rho = 1 + \frac{1}{nb}$. Then

$$\int_0^{2\pi}\left[1 + 2b(\rho - 1)\frac{\frac{b(\rho - 1)}{2} + (x\cos\varphi + y\sin\varphi)}{x^2 + y^2}\right]^{n/2} d\varphi =$$

$$= \int_0^{2\pi}\left[1 + \frac{2}{n}\frac{\frac{1}{2n} + x\cos\varphi + y\sin\varphi}{x^2 + y^2}\right]^{n/2} d\varphi = \int_0^{2\pi}\left[1 + \frac{2}{n}\cdot\Delta\right]^{\frac{n}{2}\cdot\frac{1}{\Delta}\cdot\Delta} d\varphi \le$$

$$\le \int_0^{2\pi}\exp\left(\frac{\frac{1}{2n} + x\cos\varphi + y\sin\varphi}{x^2 + y^2}\right) d\varphi =$$

$$= \exp\left(\frac{1}{2n(x^2 + y^2)}\right)\int_0^{2\pi}\exp\left(\frac{x\cos\varphi + y\sin\varphi}{x^2 + y^2}\right) d\varphi,$$

where

$$\Delta = \frac{\frac{1}{2n} + x\cos\varphi + y\sin\varphi}{x^2 + y^2}.$$

However,

$$\frac{x\cos\varphi + y\sin\varphi}{x^2 + y^2} = \frac{x}{x^2 + y^2}\cos\varphi + \frac{y}{x^2 + y^2}\sin\varphi = \cos(\gamma - \varphi) \le 1.$$

Hence

$$\left|t^{-n}P'_{2n}(t)\right| \le B\frac{1}{2\pi}n\exp\left(\frac{1}{2n|t|^2+1}\right)2\pi = \exp\left(1+\frac{1}{2n|t|^2}\right)n\|\varphi_n\|_{C(|\tau-t|=b(\rho-1))}.$$

Let us denote by $\varphi^-$ the inverse of the function $\psi$. Then we perform a conformal mapping of the exterior of $\Gamma$ onto $\{|w|>1\}$ so that $\varphi^-(\infty)=\infty$. Recall that $\varphi^-$ has the form $\varphi^-(t)=\frac{1}{c}t+\frac{\beta_0}{t}+\frac{\beta_1}{t^2}+\dots$.

Let $R>1$ and $\Gamma_R$ be a level line of mapping $\psi$ for $|w|=R$. Consider the function $\frac{\varphi_n}{(\varphi^-)^n}$. This function is analytic on $F_-$, including the point $\infty$. Therefore, by the maximum principle, we have

$$\max_{t\in D_R^-}\left|\frac{\varphi_n(t)}{\left[\varphi^-(t)\right]^n}\right| = \max_{t\in D_R^-}\frac{|\varphi_n(t)|}{|\varphi^-(t)|^n} = \max_{t\in\Gamma_R}\frac{|\varphi_n(t)|}{|\varphi^-(t)|^n} = \frac{\max_{t\in\Gamma_R}|\varphi_n(t)|}{R^n}.$$

We note that $D_R^-$ is a part of $F_-$, which is the exterior for the level line $\Gamma_R$. We also note that for $t\in\Gamma_R$ we have $|\varphi^-(t)|=R$. Since $D_R^-\subset F_-$, then

$$\max_{t\in D_R^-}\left|\frac{\varphi_n(t)}{\left[\varphi^-(t)\right]^n}\right| \le \max_{t\in F_-}\left|\frac{\varphi_n(t)}{\left[\varphi^-(t)\right]^n}\right| = \max_{t\in\Gamma}\frac{|\varphi_n(t)|}{|\varphi^-(t)|^n} = \max_{t\in\Gamma}|\varphi_n(t)|.$$

Now we use that for $t\in\Gamma$, $|\varphi^-(t)|=1$. Hence

(8) $$\max_{t\in\Gamma_R}|\varphi_n(t)| \le \max_{t\in\Gamma}|\varphi_n(t)|R^n.$$

Further, let $w=\varphi^+(t)$ be the mapping which is the interior of $\Gamma$ on $\{|w|<1\}$ such that $\varphi^+(0)=0$. It is known that $\varphi^+(t)=t+\gamma_1 t^2+\gamma_2 t^3+\dots$. Then $\left[\varphi^+\right]^n\varphi_n$ is an analytic function on $F_+$. Let $r<1$, $\Gamma_r$ be the level line of the mapping $\varphi^+(t)$ for $|w|=r$. Note that $\left[\varphi^+(t)\right]^{-1}$ is an inverse function of $\varphi^+$ and $D_r^+$ is a part of $F_+$, which is in the interior of $\Gamma_r$. Then, by the maximum principle,

$$\max_{t\in D_R^+}\left|\varphi_n(t)\left[\varphi^+(t)\right]^n\right| = \max_{t\in\Gamma_r}\left\{|\varphi_n(t)|\left|\varphi^+(t)\right|^n\right\} = \|\varphi_n\|_{C(\Gamma_r)}\cdot r^n.$$



Since $D_r^+\subset F^+$, then

$$\max_{t\in D_r^+}\left|\varphi_n(t)\left[\varphi^+(t)\right]^n\right| \le \max_{t\in F^+}\left|\varphi_n(t)\left[\varphi^+(t)\right]^n\right| = \max_{t\in\Gamma}\left|\varphi_n(t)\left[\varphi^+(t)\right]^n\right| = \max_{t\in\Gamma}|\varphi_n(t)|.$$

In this relation we consider that for $t\in\Gamma$, $\left|\varphi^+(t)\right|=1$. Therefore,

(9) $$\|\varphi_n\|_{C(\Gamma_r)}r^n \le \|\varphi_n\|_{C(\Gamma)}.$$

The function $\varphi_n$ is analytic in the ring $F$ bounded by $\Gamma_r$ and $\Gamma_R$. Hence, by the maximum principle and by inequalities (8) and (9), it follows that

$$\max_{t\in F}|\varphi_n(t)| = \max\left\{\|\varphi_n\|_{C(\Gamma_r)};\|\varphi_n\|_{C(\Gamma_R)}\right\} \le \max\left\{\|\varphi_n\|_{C(\Gamma)}\frac{1}{r^n};\|\varphi_n\|_{C(\Gamma)}R^n\right\}.$$

Let $r=1-\frac{1}{n}$ and $R=1+\frac{1}{n}$. Then $R^n<e$ and $r^n>\frac{1}{4}$.

So $\|\varphi_n\|_{C(F)}\le 4\|\varphi_n\|_{C(\Gamma)}$ and, finally, $\|\varphi'_n\|_{C(\Gamma)}\le\mu_6 n\|\varphi_n\|_{C(\Gamma)}$, where

$$\mu_6=\max_{t\in\Gamma}\left\{\frac{1}{|t|}+4\exp\left(1+\frac{1}{2n|t|^2}\right)\right\}\le\left(d+4\exp\left(1+\frac{d^2}{2}\right)\right),\ d=\frac{1}{\min|t|},\ t\in\Gamma.$$

THEOREM 2. *Let $\varphi_n\in X_n$. Then*

(10) $$\|\varphi_n\|_\beta\le\mu_7 n^\beta\|\varphi_n\|_C,\quad 0<\beta\le 1.$$

*Proof.* First assume that $|t_1-t_2|\ge\frac{1}{n}$, $(t_1,t_2\in\Gamma)$. Then

$$\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|^\beta}\le 2n^\beta\|\varphi_n\|_C.$$

Now consider that $|t_1-t_2|<\frac{1}{n}$. In this case, from

$$|\varphi_n(t_1)-\varphi_n(t_2)|=\left|\int_{t_1}^{t_2}\varphi'_n(\tau)d\tau\right|\le\|\varphi'_n\|_C\cdot lengtht_1t_2\le const\|\varphi'_n\|_C|t_1-t_2|$$

and Lemma 1, we have

$$\frac{|\varphi_n(t_1)-\varphi_n(t_2)|}{|t_1-t_2|^{\beta}} \le const\cdot\mu_6\cdot n\|\varphi_n\|_C\,|t_1-t_2|^{1-\beta} \le const\cdot\mu_6\cdot n^{\beta}\|\varphi_n\|_C .$$

From these cases we deduce (10), where $\mu_7 = \max(1+const\cdot\mu_6;3)$.

Further, we establish the estimation relation of the interpolating Lagrange polynomial for the function $\varphi$ defined on $\Gamma$.

LEMMA 2. *Let $\varphi\in H_\alpha(\Gamma)$ and $\varphi_n\in X_n$ be a polynomial such that*

$$\|\varphi-\varphi_n\|_C \le (\mu_8+\mu_9\ln n)E_n(\varphi).$$

*Then*

$$\|\varphi-\varphi_n\|_\beta \le \frac{(\mu_{10}+\mu_{11}\ln n)}{n^{\alpha-\beta}}H(\varphi;\alpha) \quad (0<\beta\le\alpha\le 1). \tag{11}$$

*Proof.* Let $\varphi_n^*$ be the polynomial of the best approximation for the function $\varphi$. From (5) and (10) it follows that

$$\|\varphi-\varphi_n\|_\beta \le \|\varphi-\varphi_n^*\|_\beta+\|\varphi_n^*-\varphi_n\|_\beta \le \mu_3\frac{1}{n^{\alpha-\beta}}H(\varphi;\alpha)+\mu_7 n^{\beta}\|\varphi_n^*-\varphi_n\|_C .$$

Using (4) for the second term of the last inequality, we deduce that

$$\|\varphi_n^*-\varphi_n\|_C \le \|\varphi_n^*-\varphi\|_C+\|\varphi_n-\varphi\|_C \le \mu_2\frac{1}{n^{\alpha}}H(\varphi;\alpha)+(\mu_8+\mu_9\ln n)E_n(\varphi) \le$$

$$\le (1+\mu_8+\mu_9\ln n)\mu_2\frac{1}{n^{\alpha}}H(\varphi;\alpha) = (\mu_{12}+\mu_{13}\ln n)\frac{1}{n^{\alpha}}H(\varphi;\alpha).$$

Then for $\mu_{10}=\mu_3+\mu_7\mu_{12}$ and $\mu_{11}=\mu_7\mu_{13}$ we obtain inequality (11).

Let $\{t_j\}_{j=0}^{2n}$ be a consequence of $2n+1$ distinct points from $\Gamma$ and

$$l_j(t) = \frac{\prod\limits_{k=0;k\ne j}^{2n}(t-t_k)}{\prod\limits_{k=0;k\ne j}^{2n}(t_j-t_k)}\left(\frac{t_j}{t}\right)^n \equiv \sum_{k=-n}^{n}\Lambda_k^{(j)}t^k \quad \left(t\in\Gamma,\ j=\overline{0,2n}\right).$$

By $U_n$ we denote the operator which maps any function $g$ continuous on $\Gamma$ onto its interpolating Lagrange polynomial defined by using the nodes $\{t_j\}_{j=0}^{2n}$. This is a polynomial of the form



$$U_n(g,t) = \sum_{j=0}^{2n} l_j(t)g(t_j).$$

It is evident that $U_nX_n = X_n$. As in [5, p. 539], we establish that

$$|U_n(g,t)-g(t)| \le (1+\lambda_n)E_n(g),\quad g(t)\in C(\Gamma),$$

where

$$\lambda_n = \max_{t\in\Gamma}\sum_{j=0}^{2n}|l_j(t)|.$$

Let us consider the function $\psi$, $\psi(w)=cw+c_0w^{-1}+\ldots$, which maps conformally the exterior of the unit circle with the center 0 onto $F_-$ so that $\psi(\infty)=\infty$, $\psi(\infty)=c>0$.

Let $w_j=\exp\frac{2\pi i}{2n+1}(j-n)$ $(i^2=-1,\ j=0,1,\ldots,2n)$ be a system of equidistance points on $\Gamma_0=\{|w|=1\}$, and

$$t_j=\psi(w_j),\quad j=0,1,\ldots,2n. \tag{12}$$

THEOREM 3. *If $t_j$, $j=0,1,\ldots,2n$ are defined by (12), then*

$$\lambda_n \le \mu_{14}+\mu_{15}\ln n.$$

The proof of this theorem is very long. We will omit it.

2. Now, using our previous results, we propose a substantiation of the collocation method.

We consider SIE with the Cauchy kernel

$$c(t)\varphi(t)+\frac{d(t)}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{1}{2\pi i}\int_\Gamma K(t,\tau)\varphi(\tau)d\tau = f(t),\ t\in\Gamma, \tag{13}$$

in the Banach space $H_\beta(\Gamma)$, $0<\beta<1$. Here $c$, $d$, $K$ and $f$ are known functions in $H_\beta(\Gamma)$ and $\varphi$ is unknown.

According to the collocation method, we seek an approximate solution of (13) in the form of a polynomial

$$\varphi_n(t)=\sum_{k=-n}^{n}\alpha_k^{(n)}t^k,\ t\in\Gamma. \tag{14}$$

The unknown coefficients $\{\alpha_k^{(n)}\}_{k=-n}^{n}=\{\alpha_k\}_{k=-n}^{n}$ are found from the following system of linear equations

$$(15)\quad a(t_j)\sum_{k=0}^{n}\alpha_k t_j^k + b(t_j)\sum_{k=-n}^{-1}\alpha_k t_j^k + \sum_{k=-n}^{n}\alpha_k \frac{1}{2\pi i}\int_\Gamma K(t_j,\tau)\tau^k d\tau = f(t_j),$$

where $a(t) = c(t) + d(t)$, $b(t) = c(t) - d(t)$ and $t_j$, $j = 0, 1, \dots, 2n$ form a set of distinct points on $\Gamma$.

THEOREM 4. *Suppose the functions a, b and K (uniform with respect to both variables) belong to the space* $H_\alpha(\Gamma)$, $0 < \beta < \alpha \le 1$, *and let the following conditions hold:*

1) $a(t)b(t) \neq 0$, $t \in \Gamma$,
2) *ind* $a(t)b^{-1}(t) = 0$, $t \in \Gamma$,
3) *the kernel of the operator corresponding to the left side of* (13) *is empty.*

*In addition, let* $t_j$, $j = \overline{0, 2n}$, *be calculated according to* (12).

*Then, for sufficiently large n, the system* (15) *has a unique solution* $\left\{\alpha_k^{(n)}\right\}_{k=-n}^{n}$. *The approximate solutions* (14) *converge in the norm of* $H_\beta(\Gamma)$ *as* $n \to \infty$ *to the exact solution* $\varphi$ *of* (13), *whatever the function* $f \in H_\alpha(\Gamma)$. *For the rate of convergence, the following estimate holds*

$$\|\varphi - \varphi_n\|_\beta \le (\mu_{16} + \mu_{17}\ln n)n^{\beta-\alpha}H(\varphi;\alpha).$$

For the proof see [1–3].

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