REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome XXVI, N^{os} 1–2, 1997, pp. 29–32

NOTE ON THE PAPER OF I. MUNTEAN "ON THE METHOD OF NEAR EQUATIONS"

S. COBZAS

For two normed spaces X, Y over the field of real or complex numbers, denote by $L(X, Y)$ the space of all continuous linear operators from X to Y, and let $L(X) = L(X, X)$.

Recall, for convenience, Banach's theorems on the invertibility of perturbed operators (see [1], Theorems V.4.3 and V.4.4).

THEOREM A. *If X is a Banach space and $A \in L(X)$ is such that $\|A\| < 1$, then the operator $I - A$ is invertible and*

(1) $$\left\|(I - A)^{-1}\right\| \leq \frac{1}{1 - \|A\|}.$$

THEOREM B. *Let X be a Banach space and Y a normed space. If $S, T \in L(X, Y)$ are such that S is invertible and $\|S^{-1}T\| < 1$ then $S + T$ is invertible and*

(2) $$\left\|(S + T)^{-1}\right\| \leq \frac{\|S^{-1}\|}{1 - \|S^{-1}T\|}.$$

For a normed space X, an operator $A \in L(X)$ and an element $y \in X$, consider the equation

(3) $$(I - A)x = y.$$

Approximating the operator A by another operator $\tilde{A} \in L(X)$ and the element y by $\tilde{y} \in X$, one obtains a new equation

(4) $$(I - \tilde{A})\tilde{x} = \tilde{y},$$

1991 AMS Subject Classification: 46B28.

easier to solve and called a *near equation* to (3). The problem considered in [2] was to give estimations of the error $\|x - \tilde{x}\|$ in terms of $\|A - \tilde{A}\|$ and $\|y - \tilde{y}\|$.

The main result in [2] is the following Kantorovich-type theorem:

THEOREM 1. *Let X be a Banach space and let $A, \tilde{A} \in L(X)$ be such that $I - A$ is invertible. Suppose that α, β, γ are three nonnegative real numbers such that*

(5) $$\alpha\beta < 1$$

and

(6) $$\|(I - A)^{-1}\| \le \alpha, \ \|A - \tilde{A}\| \le \beta, \ \|y - \tilde{y}\| \le \gamma.$$

Then the operator $I - \tilde{A}$ is invertible, too, and the solutions $x, \tilde{x}$ of equations (3) *and* (4) *verify the estimations*

(7) $$\|x - \tilde{x}\| \le \alpha\gamma + \frac{\alpha^2\beta}{1 - \alpha\beta}\|\tilde{y}\|.$$

The key tool we shall use in the proof of Theorem 1 is the following

PROPOSITION 1. *Let X be a normed space and let the operators $A, \tilde{A}$ in $L(X)$ be such that $I - A$ and $I - \tilde{A}$ are invertible. Then the solutions $x, \tilde{x}$ of equations* (3) *and* (4) *verify the identities*

(8) $$x - \tilde{x} = (I - A)^{-1}(y - \tilde{y}) + (I - A)^{-1}(A - \tilde{A})(I - A)^{-1}\tilde{y}$$

and

(9) $$x - \tilde{x} = (I - \tilde{A})^{-1}(y - \tilde{y}) + (I - A)^{-1}(A - \tilde{A})(I - A)^{-1}y.$$

Proof. The identity

(10) $$U^{-1} - V^{-1} = U^{-1}(V - U)V^{-1}$$

is true for any pair U, V of invertible operators in $L(X)$.

Writing

$$x - \tilde{x} = (I - A)^{-1}y - (I - \tilde{A})^{-1}\tilde{y} =$$

$$= (I - A)^{-1}(y - \tilde{y}) + \left[(I - A)^{-1} - (I - \tilde{A})^{-1}\right]\tilde{y}$$

and applying formula (10) to $U = I - A$ and $V = I - \tilde{A}$, we get (8).

A similar argument applied to

$$x - \tilde{x} = (I - \tilde{A})^{-1}(y - \tilde{y}) + \left[(I - A)^{-1} - (I - \tilde{A})^{-1}\right]y$$

yields (9).

Proof of Theorem 1. By (5) and (6), we have

$$\|(I - A)^{-1}(A - \tilde{A})\| \le \|(I - A)^{-1}\|\|A - \tilde{A}\| \le \alpha\beta < 1,$$

so that we can apply Theorem B to $S = I - A$ and $T = A - \tilde{A}$, to infer that the operator $I - \tilde{A}$ is invertible and that

(11) $$\|(I - \tilde{A})^{-1}\| \le \frac{\|(I - A)^{-1}\|}{\|A - \tilde{A}\|} \le \frac{\alpha}{1 - \alpha\beta}.$$

Now, equality (8) and inequalities (5) and (6) yield

$$\|x - \tilde{x}\| \le \|(I - A)^{-1}\|\|y - \tilde{y}\| + \|(I - A)^{-1}\|\|A - \tilde{A}\|\|(I - \tilde{A})^{-1}\|\|\tilde{y}\| \le$$

$$\le \alpha\gamma + \alpha\beta\frac{\alpha}{1 - \alpha\beta}\|\tilde{y}\|,$$

i.e., (7) holds.

Remark. Starting with (9) and taking into account inequality (11), one obtains the delimitation

(12) $$\|x - \tilde{x}\| \le \frac{\alpha}{1 - \alpha\beta}\gamma + \alpha\beta\frac{\alpha}{1 - \alpha\beta}\|y\|.$$

Some variations on the theme of near equations are presented in the following proposition (compare to [2, Theorem 3.1]).

PROPOSITION 2. *Let X be a Banach space, $A, \tilde{A} \in L(X)$ and $y, \tilde{y} \in X$. Suppose that p, q, r are nonnegative numbers such that*

(13) $$p + q < 1$$

and

(14) $$\|A\| \le p, \ \|A - \tilde{A}\| \le q, \ \|y - \tilde{y}\| \le r.$$

It follows that the operators $I - A$ and $I - \tilde{A}$ are invertible and the following estimations

(15) $$\|x - \tilde{x}\| \le \frac{1}{(1 - p)(1 - p - q)}\left[r(1 - p - q) + q\|\tilde{y}\|\right]$$

and

(16) $$\|x - \tilde{x}\| \le \frac{1}{(1 - p)(1 - p - q)}\left[r(1 - p) + q\|y\|\right]$$

hold.

Proof. By (13) and (14), $\|A\| \leq p < 1$, so that, by Theorem A, the operator $I - A$ is invertible and

$$\left\|(I - A)^{-1}\right\| \leq 1 / (1 - p). \tag{17}$$

Again by (13) and (14) we have

$$\left\|(I - A)^{-1}(A - \widetilde{A})\right\| \leq q / (1 - p) < 1, \tag{18}$$

so that, by Theorem B, the operator $I - \widetilde{A} = (I - A) + (A - \widetilde{A})$ is invertible, too, and

$$\left\|(I - \widetilde{A})^{-1}\right\| \leq \frac{\left\|(I - A)^{-1}\right\|}{1 - \left\|(I - A)^{-1}(A - \widetilde{A})\right\|},$$

which yields

$$\left\|(I - \widetilde{A})^{-1}\right\| \leq 1 / (1 - p - q). \tag{19}$$

Now, using equality (8) and inequalities (17) and (19), we obtain

$$\|x - \widetilde{x}\| \leq (1 - p)^{-1} r + (1 - p)^{-1} q (1 - p - q)^{-1} \|\widetilde{y}\|,$$

which is equivalent to (15).

Similarly, starting with (9) and applying again inequalities (17) and (19), one obtains the delimitation (16).

REFERENCES

1. L. V. Kantorovich and G. P. Akilov, *Functional Analysis,* Third Edition, Nauka, Moscow, 1984 (in Russian).
2. I. Muntean, *On the method of near equations*, CALCOLO (in print).

Received October 15, 1996

Faculty of Mathematics
"Babeş-Bolyai" University
RO-3400 Cluj-Napoca
Romania
E-mail: scobzas@math.ubbcluj.ro