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HERMITE-TYPE SHEPARD OPERATORS

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Let P_i, $P_i := (x_i, y_i)$, $i = 1, \ldots, n$ be distinct points in a plane domain D, f a real-valued function defined on D and $\mathscr{I}(f) := \{\lambda_i f | i = 1, \ldots, N\}$ a given set of information about f.

The original Shepard interpolation operator S_0 is defined by [6]

(1) $$S_0 f = \sum_{i=1}^{N} A_i f(x_i, y_i)$$

with

$$A_i(x, y) = \prod_{\substack{j=1 \\ j \neq 1}}^{N} d_j^{\mu}(x, y) / \sum_{k=1}^{N} \prod_{\substack{j=1 \\ j \neq k}}^{N} d_j^{\mu}(x, y),$$

where $d_j(x, y)$ is the distance between the point $(x, y) \in D$ and (x_i, y_i) and $\mu \in \mathbf{R}_+$. As can be seen, the information here is $\mathscr{I}(f) = \{\lambda_i f = f(x_i, y_i) | i = 1, \ldots, N\}$, i.e., a Lagrange-type information.

It is well known that S_0 is an interpolative operator

$$(S_0 f)(x_i, y_i) = f(x_i, y_i), \; i = 1, \ldots, N$$

and the degree of exactness (abbreviated by "dex") of S_0 is zero, i.e., $\text{dex}(S_0) = 0$: $S_0 f = f$ only for the constant function f. The second property follows from the relation

(2) $$\sum_{i=1}^{N} A_i = 1,$$

which is easy to verify.

A way to generalize the Shepard operator is to increase the degree of exactness,

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Such an extension can be given by using Hermite-type information about f

$$\mathscr{I}_H(f) = \left\{\lambda_k^{(p,q)} f \,\middle|\, \lambda_k^{(p,q)} f = f^{(p,q)}(x_k, y_k),\ k = 1,\ldots,n;\ p,q \in \mathbf{N},\ p+q \le s_k\right\}$$

with $|\mathscr{I}_H(f)| = N$. A similar extension was given in [1] for $s_1 = \ldots = s_n = m$

$$(S_m^T f)(x, y) = \sum_{k=1}^{n} A_k(x, y) T_m(x, y; x_i, y_i), \tag{3}$$

where $T_m f$ is the Taylor polynomial for the node (x_i, y_i)

$$(T_m f)(x, y; x_i, y_i) = \sum_{p+q \le m} \frac{(x-x_i)^p}{p!} \frac{(y-y_i)^q}{q!} f^{(p,q)}(x_i, y_i). \tag{4}$$

It is proved that, for $\mu \ge m$

$$(S_m f)^{(p,q)}(x_i, y_i) = f^{(p,q)}(x_i, y_i),\quad p, q \in \mathbf{N},\ p+q \le m,$$

for all $i = 1,\ldots, n$ and

$$\operatorname{dex}(S_m f) = m.$$

Remark 1. The operator S_1 was given by Shepard [6].

The so-called Shepard-Taylor operator S_m^T has the degree of exactness m. But it involves as information the values of the partial derivatives of the function f up to the order m, that make its applications difficult for large m.

In a recent paper [3], there have been studied some Lagrange-type Shepard operators, i.e., Shepard operators which use Lagrange-type information, as S_0 from (1) does, but with the degree of exactness dex > 0.

Our goal here is to construct Hermite-type Shepard operators, with the degree of exactness dex > 0.

DEFINITION 1. A Shepard operator is called of the Hermite type if the information used, $\mathscr{I}(f)$ is of the Hermite type.

One denotes by $\mathscr{I}_H^s(f)$ the information $\mathscr{I}_H(f)$ for $s_1 = \ldots = s_n = s$ (at each point P_i we have the same information about f).

Now, to each point P_i it is associated a Hermite-type interpolation polynomial of the degree m, based on some information about f at $P_i, P_{i+1}, \ldots, P_{i+\nu}$ with $\nu < n$. Let $H_i^m f$ for $i = 1,\ldots, n$, be these polynomials, if they exist. Then

$$S_m^H f = \sum_{i=1}^{n} A_i H_i^m f \tag{5}$$

is the corresponding Hermite-type Shepard operator.

Remark 2. Taking into account that [1]

$$A_i(x_k, y_k) = \delta_{ik},\ i, k = 1,\ldots,n$$

$$A_i^{(p,q)}(x_k, y_k) = 0,\ k = 1,\ldots,n;\ 1 \le p+q \le m, \text{ for } i = 1,\ldots,n,$$

it follows that $S_m^H f$ preserves all the interpolation properties of the polynomials $H_i^m f$ for $i = 1,\ldots, n$. Also, relation (2) implies that the exactness degree of the Shepard operator S_m^H is equal to the exactness degree of the Hermite operator H_i^m, i.e., $\operatorname{dex}(S_m^H) = m$.

The difficult problem here is to prove the existence and the uniqueness of the Hermite operators H_i^m, $i = 1,\ldots, n$. A first difficulty is that a two-variate polynomial of the degree m contains $(m+1)(m+2)/2$ arbitrary coefficients. This means that the number of the interpolation conditions (the cardinality of the information set) must be $(m+1)(m+2)/2$. A second difficulty arises in solving the linear system of order $(m+1)(m+2)/2 \times (m+1)(m+2)/2$, determined by the interpolation conditions. As the points $P_i \in D$ are arbitrarily distributed, there is no guarantee that the corresponding system has a unique solution. To pass these difficulties we must take a set of $(m+1)(m+2)/2$ information and organize the interpolation points P_i, $i = 1,\ldots, n$ in such sets $\{P_i, \ldots, P_{i+m}\}$, $i = 1,\ldots, n$, that each of the corresponding systems may have a unique solution.

Next we consider some concrete cases.

1. Let $\mathscr{I}(f)$ be the following set of pieces of information

$$\mathscr{I}(f) = \left\{f^{(p,q)}(x_i, y_i) \,\middle|\, i = 1,\ldots,n;\ p,q = 0,1;\ p+q \le 1\right\}. \tag{6}$$

Remark 3. For $m = 1$,

$$\mathscr{I}_k(f) = \left\{f(x_k, y_k), f^{(1,0)}(x_k, y_k), f^{(0,1)}(x_k, y_k)\right\}$$

and $H_k^1 = T_k^1$ from (4). Hence, H_k^1 exists and is unique and $S_1^H = S_1^T$ from (3).

For $m = 2$, we must take a set of six pieces of information, in order to determine the Hermite interpolation polynomials H_k^2, $k = 1,\ldots, n$, of the second degree. Let this information be

$$\mathscr{I}_k(f) := \tag{7}$$

$$:= \left\{f(x_k, y_k), f^{(1,0)}(x_k, y_k), f^{(0,1)}(x_k, y_k), f(x_{k+1}, y_{k+1}), f(x_{k+2}, y_{k+2}), f(x_{k+3}, y_{k+3}),\right\},$$

for all $k = 1,\ldots,n$, with $P_{n+1} := P_1$, $P_{n+2} := P_2$, $P_{n+3} := P_3$.

The existence and uniqueness of the polynomials $H_k^2 f$ are based on the following

LEMMA 1. *Let $Q_i := (x_i, y_i)$, $i = 0, 1, 2, 3$ be given points in the plane such that:*

a) $x_i \neq x_0$ *for* $i = 1, 2, 3$;

b) *if l_i is the line determined by the points Q_0 and Q_i then $l_j \neq l_k$ for $j, k = 1, 2, 3$, $j \neq k$. Then for every function f with the given information*

$$\mathscr{I}(f) = \left\{ f(Q_0), f^{(1,0)}(Q_0), f^{(0,1)}(Q_0), f(Q_1), f(Q_2), f(Q_3) \right\}$$

there exists a unique polynomial P_2 of the second degree that satisfies

$$(8)\qquad \begin{cases} P^{(j,k)}(Q_0) = f^{(j,k)}(Q_0), & j, k = 0, 1;\ j + k \leq 1, \\ P(Q_i) = f(Q_i), & i = 1, 2, 3. \end{cases}$$

Proof. Let P_2 be an arbitrary second-degree polynomial

$$(9)\qquad P_2(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

From (8) one obtains a 6×6 algebraic linear system. Let M be its matrix. Then, after some permissible transformations, one obtains

$$\det M = -\prod_{i=1}^{3} (x_i - x_0)^2 \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{vmatrix}$$

where

$$(10)\qquad \alpha_i = \frac{y_i - y_0}{x_i - x_0}, \quad i = 1, 2, 3.$$

Conditions a) and b) from Lemma 1 imply that $\det M \neq 0$, i.e., the system has a unique solution.

THEOREM 1. *Let f be a real-valued function for which there exists the information $\mathscr{I}(f)$ from (6). If the interpolation nodes P_k, $k = 1, \ldots, n$, are such that*

a) $x_{k+i} \neq x_k$, $i = 1, 2, 3$,

b) *$l_{i+k} \neq l_{j+k}$ for $i, j = 1, 2, 3$ with $i \neq j$, when l_{i+k} is the line determined by P_k and P_{k+i}, for all $k = 1, \ldots, n$,*

then there exists the Hermite-type Shepard operator S_2^H defined by

$$S_2^H f = \sum_{k=1}^{n} A_k H_k^2 f,$$

where H_k^2 is the Hermite interpolation operator corresponding to the information $\mathscr{I}_k(f)$ from (7), and $\operatorname{dex}(S_2^H) = 2$.

The proof follows from Lemma 1 and Remark 2.

2. A second case, for $m = 2$, is obtained by using the information

$$(11)\qquad \mathscr{I}_k(f) = \left\{ f(P_k), f^{(1,0)}(P_k), f^{(0,1)}(P_k), f^{(1,1)}(P_k), f(P_{k+1}), f(P_{k+2}) \right\},$$

for all $k = 1, \ldots, n$.

LEMMA 2. *Let $Q_i := (x_i, y_i)$, $i = 0, 1, 2$, be given points in the plane such that*

$$(12)\qquad \begin{array}{l} \text{a) } x_1 \neq x_0 \text{ and } x_2 \neq x_0 \\ \text{b) } \alpha_1 \neq \alpha_2 \text{ and } \alpha_1 \neq -\alpha_2, \text{ for } \alpha_1 \text{ and } \alpha_2 \text{ given in (10).} \end{array}$$

Then for every function f with the given information

$$\mathscr{I}(f) = \left\{ f(Q_0), f^{(1,0)}(Q_0), f^{(0,1)}(Q_0), f^{(1,1)}(Q_0), f(Q_1), f(Q_2) \right\}$$

there exists a unique polynomial P_2 of the second degree that satisfies

$$(13)\qquad \begin{cases} P_2^{(j,k)}(Q_0) = f^{(j,k)}(Q_0), & (j,k) \in \{(0,0), (0,1), (1,0), (1,1)\} \\ P_2(Q_1) = f(Q_1) \\ P_2(Q_2) = f(Q_2). \end{cases}$$

Proof. Taking P_2 as in (9), we must analyze the system given by conditions (13). But the determinant of its matrix, say Δ, can be written in the form

$$\Delta = (x_1 - x_0)^2 (x_2 - x_0)^2 (\alpha_2^2 - \alpha_1^2).$$

Hence, from conditions (12), it follows that $\Delta \neq 0$.

Now it is very easy to prove the following

THEOREM 2. *Let f be a real-valued function for which there exists the information*

$$\mathscr{I}(f) = \left\{ f^{(p,q)}(P_i) \,\middle|\, i = 1, \ldots, n; (p,q) \in \{(0,0), (0,1), (1,0), (1,1)\} \right\}.$$

If the interpolation nodes P_k, $k = 1,\ldots, n$ *are such that*

a) $x_{k+i} \neq x_k$, $i = 1,2$

b) $\dfrac{y_{k+1} - y_k}{x_{k+1} - x_k} \neq \pm \dfrac{y_{k+2} - y_k}{x_{k+2} - x_k}$, *for all* $k = 1,\ldots, n$, *then there exists the Hermite-type Shepard operator* $\widetilde{S}_2^H$, *defined by*

$$\widetilde{S}_2^H f = \sum_{k=1}^{n} A_k \widetilde{H}_k^2 f,$$

where $\widetilde{H}_k^2$ *is the Hermite interpolation operator corresponding to the information* $\mathscr{I}_k(f)$ *from* (11), *with* $\operatorname{dex}\left(\widetilde{S}_2^H\right) = 2$.

Proof. Lemma 2 assured the existence and uniqueness of all Hermite operators $\widetilde{H}_k^2$, $k = 1,\ldots, n$. Now, from Remark 2, the proof follows.

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