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CONVEXITY RELATED TO THE COARSENESS OF CONCAVITY

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1. INTRODUCTION

During the meetings of the Interdisciplinary Researches Laboratory of Cluj-Napoca "Babeş-Bolyai" University between 1980 and 1983, Elena Popoviciu discussed about the significance of a notion of behaviour, establishing a connection with the property of convexity of a function and obtaining one of its generalizations in [11]. This notion was defined by the same author in 1965 and published in 1983 [11], in order to describe the situation when an object a belonging to a given set A can be transformed by means of a known transformation $T : A \to B$ into an object $T(a)$ belonging to a particular subset $D \subset B$. The pair (D, T) is said to be the behaviour of a. In [3] this type of behaviours is called strict. Elena Popoviciu also used another notion of behaviour, called in [3] a strong behaviour and defined as follows: if $\mathscr{T}$ is a set of transformations $T : A \to B$ and for every $T \in \mathscr{T}$, $T(a) \in D$ is true, then the pair $(D, \mathscr{T})$ is said to be the behaviour of the element $a \in A$. A weaker notion of behaviour was considered in [3] by means of the same type of set of transformations $\mathscr{T}$, $T : A \to B$. If for the element $a \in A$ there is a transformation $T \in \mathscr{T}$, such that $T(a) \in D$ is true, then the pair $(D, \mathscr{T})$ is called a weak behaviour of the element a. Strict behaviours, strong behaviours and weak behaviours are called behaviours in A by means of B. The set of the behaviours defined in A by means of B was denoted in [3] by Comp(A, B) and described by

$$\begin{aligned} \mathrm{Comp}(A,B) &= \{C = (D,T) \mid D \subseteq B, T : A \overset{\bullet}{\to} B\} \cup \\ (1.1)\qquad &\cup \{C = s-(D,\mathscr{T}) \mid D \subseteq B, \mathscr{T} \subset \{f \mid f : A \to B\}\} \cup \\ &\cup \{C = w-(D,\mathscr{T}) \mid D \subseteq B, \mathscr{T} \subset \{f \mid f : A \to B\}\}. \end{aligned}$$

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If $C \in \mathrm{Comp}(A, B)$, then the set of all these elements $a \in A$ such that $T(a) \in D$ (or $T(a) \in D$ for every $T \in \mathcal{T}$, or there is a transformation $T \in \mathcal{T}$, such that $T(a) \in D$) is said to be the set of the elements of *A* having the behaviour *C* and is denoted by *C*(*A*). A possibility of organizing Comp(*A*, *B*) as a preinductive semilattice with respect to the union is presented in [3]. The strict behaviours and the strong behaviours that are allures of functions have been studied by R. Precup in his thesis [12]. Two properties of convexity of a set with respect to a given set and two behaviours have been defined and studied in [4] and are quoted in Section 2 of the present paper.

In 1983 J. M. Chassery [1] defined a notion of discrete convexity, using it in cytology investigations. He was interested in detecting these images appearing on the screen of the computer, after digitization, as convex sets, according to his notion of discrete convexity. In [5] the converse problem was studied: What type of properties can a set that is transformed by means of a digitization method into a convex-like set have? The result of these investigations is the remark that these types of sets have some special type of convexity properties that are described in Section 3 of this paper. More examples are presented, showing that even some bounded fractals are convex in this manner. The aim of Sections 3 and 4 is to establish the connection between the properties defined in [4], the convexities from [5] and the notion of coarseness of the concavity defined by A. Rosenfeld in [13].

2. CONVEXITIES WITH RESPECT TO A SET AND TWO BEHAVIOURS

X is assumed to be a nonempty set so that for every two points *x*, *y* of *X* the closed straight line segment determined by these points, $\langle x, y\rangle$, is defined. Let *M* be a nonempty subset of *X*, B' and B'' two nonempty sets, $D' \subset B'$, $D'' \subset B''$ nonempty subsets, $T': X \times M \to B'$, $C' = (D', T')$ a behaviour in $X \times M$, $T'': M \times \mathcal{L}(X) \to B''$ and $C'' = (D'', T'')$ the corresponding behaviour in $M \times \mathcal{L}(X)$.

DEFINITION 2.1. i) *The set* $Y \subset X$ *is said to be slackly convex with respect to the set M and the behaviours C' and C'' if it is empty or if for every two points* $x, y \in Y$ *and for every* $t \in \langle x, y\rangle$, *if the element* $(t, a) \in C'(X \times M)$, *with* $a \in M$, *then the element* $(a, Y) \in C''(M \times \mathcal{L}(X))$.

ii) *The set* $Y \subset X$ *is said to be strongly convex with respect to the set M and the behaviours C' and C'' if it is empty or if for every two points* $x, y \in Y$ *and for every* $t \in \langle x, y\rangle$ *there is an element* $a \in M$ *such that the following implication takes place:* $(t, a) \in C'(X \times M)$ *involves that* $(a, Y) \in C''(M \times \mathcal{L}(X))$.

It is obvious that Definition 2.1 does not depend on the strictness, weakness or strongness of the behaviours C' and C''.

For various particularizations of the behaviours, more notions of convexity are obtained. It has been proved in [4] that all the convexity properties used by Liana Lupşa in solving various optimization problems ([7], [8], [9]) are convexities with respect to a given set *M* and two behaviours C_1 and C_2. Also, the convexity studied by S. V. Ovchinnikov in subsets of distributive lattices [10], in connection with the geometry of preference spaces [6], is a property of this type [4].

3. THE CONNECTION WITH THE DISCRETE CONVEXITY

In the plane $\mathbf{R}^2$, for every two points *x* and *y* the straight line segment from the Euclidean geometry is denoted by $\langle x, y\rangle$ and for a set $A \subset \mathbf{R}^2$ conv*A* means the union of all straight line segments determined by pairs of points of *A*.

Let $h>0$, $\varepsilon>0$, $\delta>0$, the set of all the grid knots having the step *h* be denoted by $\mathbf{Z}(h) = \{(ih, jh) \mid i \in \mathbf{Z}, j \in \mathbf{Z}\}$ and *d* be a distance in $\mathbf{R}^2$. Let us take $M = \mathbf{Z}(h)$, $B' = B'' = \mathbf{R}$, $D' = [0, \varepsilon]$, $D'' = [0, \delta]$. The behaviours C' and C'' are defined by means of the distance *d* as follows: $T': \mathbf{R}^2 \times \mathbf{Z}(h) \to \mathbf{R}$, $C' = ([0, \varepsilon], T')$, $T'(a, b) = d(a, b), (a, b) \in \mathbf{R}^2 \times \mathbf{Z}(h)$, $T'': \mathbf{Z}(h) \times \mathcal{L}(\mathbf{R}^2) \to \mathbf{R}$, $C'' = ([0, \delta], T'')$, $T''(x, A) = d(x, A) = \inf\{d(a, x) | a \in A\} (x, A) \in \mathbf{Z}(h) \times \mathcal{L}(\mathbf{R}^2)$. The closed ball in $\mathbf{Z}(h)$, having the center *s* and the radius ε, will be denoted by $B(s, \varepsilon) = \{m \in \mathbf{Z}(h) | d(m, s) \le \varepsilon\}$.

DEFINITION 3.1. *The set* $A \subset \mathbf{R}^2$ *is said to be slackly (strongly)* (ε, δ)-*convex with respect to* $\mathbf{Z}(h)$ *iff it is slackly (strongly) convex with respect to* $\mathbf{Z}(h)$ *and the behaviours C' and C''.*

It means that the set $A \subset \mathbf{R}^2$ is slackly (ε, δ)-convex with respect to $\mathbf{Z}(h)$ iff it is empty or if for every two points $x, y \in A$ and for every $t \in \langle x, y\rangle$, if there is an element $a \in \mathbf{Z}(h)$ such that $d(t, a) \le \varepsilon$, then $d(a, A) \le \delta$. The set $A \subset \mathbf{R}^2$ is strongly (ε, δ)-convex with respect to $\mathbf{Z}(h)$ iff it is empty or if for every two points $x, y \in A$ and for every $t \in \langle x, y\rangle$, there is an element $a \in \mathbf{Z}(h)$ such that $d(t, a) \le \varepsilon$ involves that $d(a, A) \le \delta$.

Example 3.1. The set $A = ([0,1] - \mathbf{Q}) \times ([0,1] - \mathbf{Q})$, where $\mathbf{Q}$ is the set of rational numbers, is strongly (ε, δ)-convex with respect to $\mathbf{Z}^2$ for every $\varepsilon \ge 1/2$ and $\delta > 0$, if *d* is the chessboard distance $d(a, b) = \max(|x_a - x_b|, |y_a - y_b|)$, for $a = (x_a, y_a) \in \mathbf{R}^2$ and $b = (x_b, y_b) \in \mathbf{R}^2$.

Example 3.2. By $G \subset [0,1]$ the set of Cantor is denoted and $A = G \times G$ is the fractal called *the carpet of Sierpinski*. If the chessboard distance is used, then A is strongly (1/2, 0)-convex with respect to $\mathbf{Z}^2$. In a similar way, the fractal called *the sponge of Sierpinski*, $G \times G \times G$, is strongly (1/2, 0)-convex with respect to $\mathbf{Z}^3$.

The relationship between these convexities is studied in [5]. They are not similar and none of them involves the other one. The following properties are proved in [5].

PROPOSITION 3.1 [5]. *If the set* $A \subset \mathbf{R}^2$ *is strongly (h/2, δ)-convex with respect to* $\mathbf{Z}(h)$, *for* $\delta \geq h$, *then it is slackly (h/2, δ)-convex with respect to* $\mathbf{Z}(h)$.

PROPOSITION 3.2 [5]. *If the set* $A \subset \mathbf{R}^2$ *is slackly* (ε, δ)*-convex with respect to* $\mathbf{Z}(h)$, *with* $\varepsilon \in [h/2, h[$ *and* $\delta \geq 0$, *then it is strongly* (ε, δ)*-convex with respect to* $\mathbf{Z}(h)$.

PROPOSITION 3.3 [5]. *If the set* $A \subset \mathbf{R}^2$ *is slackly (strongly)* (ε, δ')*-convex with respect to* $\mathbf{Z}(h)$, *then it is slackly (strongly)* (ε, δ'')*-convex with respect to* $\mathbf{Z}(h)$, *for* $\delta' \leq \delta''$.

PROPOSITION 3.4 [5]. *If the set* $A \subset \mathbf{R}^2$ *is strongly* (ε', δ')*-convex with respect to* $\mathbf{Z}(h)$, *then it is strongly* (ε'', δ'')*-convex with respect to* $\mathbf{Z}(h)$, *for every* $\varepsilon' \leq \varepsilon''$ *and* $\delta' \leq \delta''$.

Two general properties proving the consistence of the notions defined by means of Definition 3.1 are the following theorems. In the following properties the chessboard distance is used and the corresponding ball $B(a, r)$ is denoted by $B_\infty(a, r)$.

THEOREM 3.5. *For every bounded set* $A \subset \mathbf{R}^2$ *there are three real numbers* $h > 0, \varepsilon > 0, \delta \geq 0$, *such that A is strongly* (ε, δ)*-convex with respect to* $\mathbf{Z}(h)$.

Proof. 1) If $A = \{a\}$, then for every $h > 0, \varepsilon \geq h/2$ and $\delta \geq h$ the set A is strongly (ε, δ)-convex with respect to $\mathbf{Z}(h)$.

2) If A is not a singleton set, then let $h \geq \operatorname{diam} A = \sup\{d(x, y) | x \in A, y \in A\}$, $\varepsilon \geq h/2, \delta = h$. In that case four situations are possible:

a) There is a pair of integer numbers (i, j) such that $A \subseteq [ih, (i+1)h] \times [jh, (j+1)h]$. If $S = \{(ih, jh), (ih, (j+1)h), ((i+1)h, jh), ((i+1)h, (j+1)h)\}$, then for every $x \in A$ there is a point $y \in S$ such that $d(x, y) \leq h/2 \leq \varepsilon$. This property is particularly validated for $x \in \langle a, b \rangle$, $a \in A$, $b \in A$. In addition to that property, for every $y \in S, d(y, A) \leq h$, and therefore A is strongly (ε, h)-convex with respect to $\mathbf{Z}(h)$.

b) There is a pair of integer numbers (i, j) such that $A \subseteq [(i-1)h, (i+1)h] \times [jh, (j+1)h]$ (a similar situation is $A \subseteq [ih, (i+1)h] \times [(j-1)h, (j+1)h]$).

Then the reasoning from a) remains valid if S is replaced by $T = S \cup \{((i-1)h, jh), ((i-1)h, (j+1)h)\}$.

c) There is $(i, j) \in \mathbf{Z}^2$ such that $A \subseteq \{[(i-1)h, (i+1)h] \times [jh, (j+1)h]\} \cup \cup \{[ih, (i+1)h] \times [(j-1)h, jh]\}$ (or all the situations when three grid squares having a similar position are considered). Then the reasoning from a) remains valid if S is replaced by $V = T \cup \{(ih, (j-1)h), ((i+1)h, (j-1)h)\}$ and $h/2 < \varepsilon \leq h$.

d) If there is a point $y \in \mathbf{Z}(h)$ such that $y \in \operatorname{int}(\operatorname{conv} A)$, then $A \subseteq \operatorname{conv} B_\infty(y, h)$. Therefore, for every $x \in A$ there is a point $y \in B_\infty(y, h)$ such that $d(x, y) \leq h/2 \leq \varepsilon$ and, using the previous reasoning, the expected result is obtained. □

THEOREM 3.6. *For every bounded set* $A \subset \mathbf{R}^2$ *there are three real numbers* $h > 0, \varepsilon > 0, \delta \geq 0$, *such that A is slackly* (ε, δ)*-convex with respect to* $\mathbf{Z}(h)$.

Proof. 1) If $A = \{a\}$, then let $h > 0$, $\varepsilon \leq d(a, \mathbf{Z}(h))$, $\delta \geq \varepsilon$ and the result follows.

2) If A is not singleton, then $h \geq \operatorname{diam} A$ is considered. The proof is similar with that of the previous theorem for $0 < \varepsilon \leq h$ and $\delta \geq \sup\{d(x, A) | x \in S\}$. □

The first connection between the notion of discrete convexity defined by J. M. Chassery in 1983 and that from Definition 3.1 is:

DEFINITION 3.2 [1]. *A set* $A \subset \mathbf{Z}(h)$ *is said to be discretely convex if for every* $x, y \in A$ *and* $t \in [0,1]$, *there are a number* $\varepsilon \in [h/2, h]$ *and a point* $a \in \mathbf{Z}(h)$ *such that* $tx + (1-t)y \in B_\infty(a, \varepsilon)$.

The following result is now obvious:

COROLLARY 3.7. *A set* $A \subset \mathbf{Z}(h)$ *is discretely convex iff there exists a real number* $\varepsilon \in [h/2, h]$ *such that A is strongly* (ε, 0)*-convex with respect to* $\mathbf{Z}(h)$.

It is obvious that the property of strong (ε, 0)-convexity with respect to $\mathbf{Z}(h)$ is more general than the discrete convexity. This is proved by means of Examples 3.1 and 3.2. But there is an important connection between these convexities. To obtain it, it is necessary to define a method of digitization in $\mathbf{R}^2$ with respect to $\mathbf{Z}(h)$.

DEFINITION 3.3. *A function* $f : \mathbf{R}^2 \to \mathbf{Z}(h)$ *such that for every* $x \in \mathbf{Z}(h)$, $f(x) = x$, *is said to be a method of digitization of* $\mathbf{R}^2$ *with respect to* $\mathbf{Z}(h)$.

THEOREM 3.8. *If a set* $A \subset \mathbf{R}^2$ *is both strongly and slackly* (ε, 0)*-convex with respect to* $\mathbf{Z}(h)$, *for* $\varepsilon \in [h/2, h]$, *then there is a method of digitization* $f : \mathbf{R}^2 \to \mathbf{Z}(h)$ *such that f(A) is discretely convex.*

Proof. Consider the method of digitization defined by $f(a)=(x,\ y)$ if $a \in [x-h/2, x+h/2] \times [y-h/2, y+h/2]$ for every $(x,y) \in \mathbf{Z}(h)$ and $a \in \mathbf{R}^2$. Suppose that A is both strongly and slackly (ε, 0)-convex with respect to $\mathbf{Z}(h)$. Then for every $x, y \in A$ and $t \in [0,1]$ there is an $a \in \mathbf{Z}(h)$ such that $d(tx+(1-t)y, a) \le \varepsilon$ involves that $a \in A$. To prove that $f(A)$ is discretely convex, let us suppose that x and y are two points of $f(A)$. Two situations are possible:

1) $x \in A \cap f(A)$ and $y \in A \cap f(A)$;
2) x or y belongs to $f(A)-A$.

1) If $x, y \in A \cap f(A)$, then from the strong (ε, 0)-convexity of A with respect to $\mathbf{Z}(h)$ it follows that for every $t \in [0,1]$ there is an $a \in \mathbf{Z}(h)$ such that $d(tx+(1-t)y, a) \le \varepsilon$ involves that $a \in A$. But $a \in A$ and $a \in \mathbf{Z}^2(h)$ mean that $a \in f(A)$ and, therefore, the definition of the discrete convexity is fulfilled.

2) Now $x \in f(A)-A$ and $y \in A$ is supposed. Then two situations are possible in the neighbourhood of x:

a) there is $z \in A$ such that $d(x,z) < h/2$;
b) the closest point $z \in A$ satisfies $d(x,z) = h/2$.

a) If $z \in A$ such that $d(x,z) < h/2$, then there is a point $u \in \langle x, y\rangle$ satisfying $\langle z, u\rangle \subset \text{conv}B_\infty(z, h/2)$ and $\langle u, y\rangle \not\subset \text{conv}B_\infty(x, h/2)$. If $t \in [0,1]$ is chosen such that $tz+(1-t)y \in \langle z, u\rangle$, then $d(tz+(1-t)y, x) < h/2 \le \varepsilon$. Therefore, from the slack (ε, 0)-convexity of A with respect to $\mathbf{Z}(h)$ it follows that $x \in A$ and the hypothesis of the situation 2) cannot take place.

b) The slack (ε, 0)-convexity of A with respect to $\mathbf{Z}(h)$ has the consequence that $x \in A$, by taking $t=0$ in the reasoning above. Therefore, the hypothesis of the situation 2) cannot take place.

The cases $x \in f(A), y \in f(A)-A$ and $x \in f(A)-A, y \in f(A)-A$ are similar. □

Two important consequences of the proof of Theorem 3.8 must be registered, also using the chessboard distance.

COROLLARY 3.9. *If the set* $A \subset \mathbf{R}^2$ *is both strongly and slackly* (ε, 0)-*convex with respect to* $\mathbf{Z}(h)$, *with* $\varepsilon \in [h/2, h]$, *then there is a method of digitization* $f: \mathbf{R}^2 \to \mathbf{Z}(h)$ *such that* $f(A) \subseteq A$.

COROLLARY 3.10. *If the set* $A \subset \mathbf{R}^2$ *is slackly* (ε, 0)-*convex with respect to* $\mathbf{Z}(h)$, *with* $\varepsilon \in [h/2, h[$, *then it is strongly* (ε, 0)-*convex with respect to* $\mathbf{Z}(h)$.

But if $\varepsilon = h/2$ and $\delta > \varepsilon$, then the situation from Corollary 3.10 does not take place.

PROPOSITION 3.11. *If the set* $A \subset \mathbf{R}^2$ *is strongly* (h/2, δ)-*convex with respect to* $\mathbf{Z}(h)$, *for* $\delta \ge h$, *then it is slackly* (h/2, δ)-*convex with respect to* $\mathbf{Z}(h)$.

Proof. If $x, y \in A$, and $t \in \langle x, y\rangle$, then there is a point $a \in \mathbf{Z}(h)$ such that $d(t,a) \le h/2$ involves that $d(a, A) \le \delta$. But if $B = \{b \in \mathbf{Z}(h) | b \notin \text{conv}A\}$, then $\min\{d(b, \text{conv}A) | b \in B\} \le h$ and for every point $z \in \text{conv}A \cap \mathbf{Z}(h)$ there is a point $u \in B$ such that $d(z,u) = h$. Therefore, for $t \in \langle x, y\rangle$ and for every point $c \in \mathbf{Z}(h)$ such that $d(t,c) \le h/2$, $d(c, A) \le h \le \delta$ is valid and the slack (h/2, δ)-convexity of A with respect to $\mathbf{Z}(h)$ is proved. □

Proposition 3.11 is not valid for $\delta < h$. Also, generally, the convexity property of a set in $\mathbf{R}^2$ with respect to $\mathbf{Z}(h)$ is not valid if another grid, having the step $h' < h$, is considered. Both remarks are proved by the following example.

Example 3.3. In $\mathbf{Z}^2$ the set $A = \{(0, 0), (0, 1), (1, 0), (2, 0), (3, 0)\}$ is taken into account. This set is strongly (1/2, 1)-convex with respect to $\mathbf{Z}^2$, it is also strongly (1/4, 1/2)-convex with respect to $\mathbf{Z}(1/2)$, but it is not strongly (1/6, 1/3)-convex with respect to $\mathbf{Z}(1/3)$. Indeed, if $x = (0, 1)$, $y = (3, 0)$, $t = (1, 2/3) \in \langle x, y\rangle$ and $t \in \mathbf{Z}(1/3)$ but $d(t, A) = 2/3 > 1/3$.

DEFINITION 3.4. *For a set* $A \subset \mathbf{R}^2$, *the largest* $h > 0$ *such that* A *is not strongly* (h/2, h)-*convex with respect to* $\mathbf{Z}(h)$ *is called the detectability of the concavity of* A.

Now, a method used in [13] to discuss the detectability of a concavity of a set will be followed in order to suggest the significance of δ. A. Rosenfeld defined in [13] the notion of concavity coarseness of a set as follows: Let G be a grid of mesh h. For any region R he defines the G-digital image R_G^* of R as the union of all the grid squares whose intersections with R have areas at least $h^2/2$. A union of grid squares A is said to be G-convex [13] if there is a convex region R such that $R_G^* = A$. If no such R exists, then A is said to be G-concave.

A region R is said to be h-concave [13] if there is a grid G of mesh h such that R_G^* is G-concave. Then the *concavity coarseness* of R is the largest h such that R is h-concave [13].

Example 3.4. For $m \ge 2$, consider the astroid together with its inner points

$$A = \left\{ (x,y) \in \mathbf{R}^2 \left| -\left(m^{\frac{2}{3}} - x^{\frac{2}{3}} \right)^{\frac{3}{2}} \le y \le \left(m^{\frac{2}{3}} - x^{\frac{2}{3}} \right)^{\frac{3}{2}}, -m \le x \le m \right. \right\}.$$

The set A is strongly convex with respect to $\mathbf{Z}^2$ if $\varepsilon \geq m/4$ and $\delta > \frac{m(\sqrt{2}-1)}{\sqrt{2}}$.

For this set A the number $\Delta(A) = \frac{m(\sqrt{2}-1)}{\sqrt{2}}$ represents the detectability of concavity of A. This means that it is possible to find a network and a digitization of A enabling us to insert a white square (ground point) between two coloured squares (image points) if the grid step is at most $\Delta(A) = \frac{m(\sqrt{2}-1)}{\sqrt{2}}$. To prove that, the geometric significance of this convexity property is used.

4. THE GEOMETRIC CHARACTERIZATION

In what follows, for a point x from $\mathbf{R}^2$ and a nonnegative real number n the notation $B_\infty(x, n) = \mathrm{conv}B_\infty(x, n)$ will be used for simplification.

THEOREM 4.1. *The set* $A \subset \mathbf{R}^2$ *is slackly* (ε, δ)*-convex with respect to* $\mathbf{Z}(h)$ *iff for every x and y from A and for every* $a \in \mathrm{conv}(B_\infty(x,\varepsilon) \cup B_\infty(y,\varepsilon)) \cap \mathbf{Z}(h)$ *the set* $B_\infty(a,\delta) \cap A$ *is not empty.*

Proof. Let us suppose that A is slackly (ε, δ)-convex with respect to $\mathbf{Z}(h)$. Then, if x and y are two points of A and $t \in \langle x, y \rangle$, the condition $d(a,t) \leq \varepsilon$ for $a \in \mathbf{Z}(h)$ means that $a \in \mathrm{conv}(B_\infty(x,\varepsilon) \cup B_\infty(y,\varepsilon)) \cap \mathbf{Z}(h)$. But for every a satisfying this condition, the slack (ε, δ)-convexity of A has as a consequence the property that $d(a, A) \leq \delta$. This means that $B_\infty(a,\delta) \cap A \neq \varnothing$.

Now, let us suppose the converse property. If x, y belong to A and $a \in \mathrm{conv}(B_\infty(x,\varepsilon) \cup B_\infty(y,\varepsilon)) \cap \mathbf{Z}(h)$, then this means that $d(a, \langle x, y \rangle) \leq \varepsilon$ and, therefore, $B_\infty(a,\varepsilon) \cap \langle x, y \rangle \neq \varnothing$. Let us consider the set $V(\varepsilon;\ x,\ y)\ =\ \cup \cup\{B_\infty(a,\varepsilon) \cap \langle x, y \rangle | a \in \mathrm{conv}(B_\infty(x,\varepsilon) \cup B_\infty(y,\varepsilon)) \cap \mathbf{Z}(h)\}$. Then for every $t \in V(\varepsilon; x, y)$ there is $a \in \mathbf{Z}(h)$ such that $d(t,a) \leq \varepsilon$ and for these points a the condition $B_\infty(a,\delta) \cap A \neq \varnothing$ means that $d(a, A) \leq \delta$. If $t \notin V(\varepsilon; x, y)$ then for every $a \in \mathbf{Z}(h)$, $d(t,a) > \varepsilon$. Then the definition of the slack (ε, δ)-convexity of A with respect to $\mathbf{Z}(h)$ is fulfilled. □

Let us denote, for every two points x and y in the plane and for every real number $\varepsilon > 0$, $\mathrm{Seg}(\varepsilon; x, y) = \mathbf{Z}(h) \cap \mathrm{conv}(B_\infty(x,\varepsilon) \cup B_\infty(y,\varepsilon))$.

THEOREM 4.2. *The set* $A \subset \mathbf{R}^2$ *is both slackly and strongly* (ε, δ)*-convex with respect to* $\mathbf{Z}(h)$ *iff for every* $x, y \in A$ *the following conditions are satisfied:*

1) $\langle x, y \rangle \subset \cup\{B_\infty(a,\varepsilon) \mid a \in \mathrm{Seg}(\varepsilon; x, y)\}$;

2) $a \in \mathrm{Seg}(\varepsilon; x, y) \Rightarrow B_\infty(a,\delta) \cap A \neq \varnothing$.

Proof. The condition 1) means that for every $x, y \in A$ the straight line segment $\langle x, y \rangle$ is covered by ε-neighbourhoods centred in grid knots. Therefore, the existence condition from the definition of the strong (ε, δ)-convexity with respect to $\mathbf{Z}(h)$ is satisfied. Now, the proof follows a similar route as that of the previous theorem. □

According to this geometric characterization of the convexities defined by means of Definition 3.1, we are able to find the detectability of the concavity of the set from Example 3.4. Indeed, if $x = (m, 0)$ and $y = (0, m)$, then, considering the grid $\mathbf{Z}(m/4)$, it is easy to see that it is possible to draw a square having the centre $t = (m/2, m/2)$ and the length of its side equal to $\delta < \Delta(A)$ such that this square contains only points that do not belong to A. But this square is situated between the same type of squares containing points of A. Therefore, applying a grid of step δ on $\mathbf{R}^2$, the image of A by the digitization method used in the proof of Theorem 3.8 will not be discretely convex. Therefore, for every grid having the step $p < \delta$ the concavity of the astroid is detected, and $\Delta(A)$ is the detectability of the concavity of A.

Now, if the digitization method used in [13] is replaced by that described in the proof of Theorem 3.8, then it is not difficult to prove that the detectability of the concavity of A is exactly the concavity coarseness of A and the connection with the number δ involved in the (ε, δ)-convexities presented above is obvious.

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