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CAUCHY STRUCTURES AND CONTIGUITIES

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A *screen* (= filter merotopy) on a set $X \neq \emptyset$ is a collection $\underset{\sim}{S}$ of filters in X such that

S_1 $\quad x \in X$ implies $\dot{x} \in \underset{\sim}{S}$,

S_2 $\quad \underset{\sim}{s} \in \underset{\sim}{S},\ \underset{\sim}{s} \subset \underset{\sim}{s}' \in \mathrm{Fil}\, X$ imply $\underset{\sim}{s}' \in \underset{\sim}{S}$.

$\underset{\sim}{S}$ is a *Cauchy structure* iff further

S_3 $\quad \underset{\sim}{s}, \underset{\sim}{s}' \in \underset{\sim}{S},\ \underset{\sim}{s} \Delta \underset{\sim}{s}'$ imply $\underset{\sim}{s} \cap \underset{\sim}{s}' \in \underset{\sim}{S}$

$\left(\underset{\sim}{a} \Delta \underset{\sim}{b}\right.$ means $A \cap B \neq \emptyset$ for $\left. A \in \underset{\sim}{a},\ B \in \underset{\sim}{b}\right)$.

It is well-known that a *contiguity* on X may be defined as a family $\underset{\sim}{R} \subset \Phi(X)$ (= the collection of all finite subsets of exp X) such that

R_1 $\quad \emptyset \in \underset{\sim}{r}$ implies $\underset{\sim}{r} \in \underset{\sim}{R}$ $\left(\underset{\sim}{r} \in \Phi(X)\right)$,

R_2 $\quad \underset{\sim}{r} \in \underset{\sim}{R}$ implies $\bigcap \underset{\sim}{r} = \emptyset$,

R_3 $\quad \underset{\sim}{r} \in \underset{\sim}{R},\ \underset{\sim}{r} \ll \underset{\sim}{r}' \in \Phi(X)$ imply $\underset{\sim}{r}' \in \underset{\sim}{R}$,

R_4 $\quad \{R_0, R_1, \ldots, R_s\},\ \{R_0', R_1, \ldots, R_s\} \in \underset{\sim}{R}$ imply $\{R_0 \cup R_0', R_1, \ldots, R_s\} \in \underset{\sim}{R}$

$\left(\underset{\sim}{r} \ll \underset{\sim}{r}'\right.$ iff $R \in \underset{\sim}{r}$ implies the existence of $R' \in \underset{\sim}{r}'$ with $\left. R \supset R'\right)$.

For $2 \leqq m \in \mathbf{N}$, the definition of an *m-contiguity* is obtained if $\Phi(X)$ is replaced by $\Phi_m(X) = \left\{\underset{\sim}{r} \in \Phi(X) : |\underset{\sim}{r}| \leqq m\right\}$. For $m = 2$, we obtain the concept of a Čech proximity.

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Let $2 \leqq m < n \in \mathbf{N}$. If $\underset{\sim}{R}$ is a contiguity or an n-contiguity, then ${}^{m}\underset{\sim}{R} = \underset{\sim}{R} \cap \Phi_m(X)$ is the m-contiguity *induced* by $\underset{\sim}{R}$. Conversely, if $\underset{\sim}{M}$ is an m-contiguity, there exist a coarsest (= smallest) contiguity $\underset{\sim}{R}^0(\underset{\sim}{M})$ and a coarsest n-contiguity $\underset{\sim}{N}^0(\underset{\sim}{M})$ inducing $\underset{\sim}{M}$.

If $\underset{\sim}{S}$ is a screen, then the systems $\underset{\sim}{r} \in \Phi(X)\left(\underset{\sim}{r} \in \Phi_m(X)\right)$ such that $\underset{\sim}{r}\, \Delta\, \underset{\sim}{s}$ does not hold for any $\underset{\sim}{s} \in \underset{\sim}{S}$ constitute the contiguity (m-contiguity) ${}^{\omega}\underset{\sim}{S}({}^{m}\underset{\sim}{S})$ *induced* by $\underset{\sim}{S}$. Conversely, if $\underset{\sim}{R}$ is a contiguity or an m-contiguity, there exists a coarsest (= largest) screen $\underset{\sim}{S}^0(\underset{\sim}{R})$ inducing $\underset{\sim}{R}$.

We look for conditions on $\underset{\sim}{R}$ implying that $\underset{\sim}{S}^0(\underset{\sim}{R})$ is a Cauchy structure.

If $\underset{\sim}{R}$ is a contiguity or an m-contiguity on X and $\underset{\sim}{r} \in \underset{\sim}{R}$, then an $\underset{\sim}{R}$*-swelling* of $\underset{\sim}{r}$ is a mapping $\sigma : \underset{\sim}{r} \to \exp X$ such that $\{R, X - \sigma(R)\} \in \underset{\sim}{R}$ for $R \in \underset{\sim}{r}$. We say that σ is *free* iff $\bigcap \sigma(\underset{\sim}{r}) = \varnothing$, and it is ($\underset{\sim}{R}$-) *loose* iff $\sigma(\underset{\sim}{r}) \in \underset{\sim}{R}$. The ($m$-) contiguity $\underset{\sim}{R}$ is said to be *Efremovich* iff each $\underset{\sim}{r} \in \underset{\sim}{R}$ admits a free $\underset{\sim}{R}$-swelling; it is *uniform* iff the same holds with a loose $\underset{\sim}{R}$-swelling. For $m = 2$, Efremovich m-contiguities coincide with Efremovich proximities, and in the case of a contiguity $\underset{\sim}{R}$, it is uniform iff $\left\{\{X - R : R \in \underset{\sim}{r}\} : \underset{\sim}{r} \in \underset{\sim}{R}\right\}$ is the collection of all finite coverings in a uniformity.

THEOREM 1. *For a contiguity $\underset{\sim}{R}$, the following statements are equivalent:*

a) *$\underset{\sim}{S}^0(\underset{\sim}{R})$ is a Cauchy structure.*
b) *$\underset{\sim}{R}$ is Efremovich.*
c) *$\underset{\sim}{R}$ is uniform.*
d) *For each $2 \leqq n \in \mathbf{N}$, $\underset{\sim}{N} = {}^{n}\underset{\sim}{R}$ is uniform and $\underset{\sim}{R} = \underset{\sim}{R}^0(\underset{\sim}{N})$.*
e) *The statement in* d) *holds for a single n.*
f) *If $\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_s$ and $\underset{\sim}{v}$ are ultrafilters in X such that $U \in \underset{\sim}{u}_i$, $V \in \underset{\sim}{v}$ imply $\{U, V\} \notin \underset{\sim}{R}$ $(i = 1, \ldots, s)$, then $U_i \in \underset{\sim}{u}_i$ implies $\{U_1, \ldots, U_s\} \notin \underset{\sim}{R}$.*

THEOREM 2. *For an n-contiguity $\underset{\sim}{N}\left(2 \leqq n \in \mathbf{N}\right)$, the following statements are equivalent:*

a) *$\underset{\sim}{S}^0(\underset{\sim}{N})$ is a Cauchy structure.*
b) *$\underset{\sim}{N}$ is Efremovich.*

c) *$\underset{\sim}{N}$ is uniform.*
d) *$\underset{\sim}{M} = {}^{2}\underset{\sim}{N}$ is uniform and $\underset{\sim}{N} = \underset{\sim}{N}^0(\underset{\sim}{M})$.*
e) *$\underset{\sim}{R}^0(\underset{\sim}{N})$ is uniform.*
f) *If $\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_s$ $\left(2 \leqq s \leqq n\right)$ and $\underset{\sim}{v}$ are ultrafilters such that $U \in \underset{\sim}{u}_i$, $V \in \underset{\sim}{v}$ imply $\{U, V\} \notin \underset{\sim}{N}$ $(i = 1, \ldots, s)$, then $U_i \in \underset{\sim}{u}_i$ implies $\{U_1, \ldots, U_s\} \notin \underset{\sim}{N}$.*

For $n = 2$, a) $\Leftrightarrow$ b) $\Leftrightarrow$ f) is contained in [1] and b) $\Leftrightarrow$ f) in [2]. A detailed version of this paper is in print in *Acta Math. Hung.*

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