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# ON (*h, k*)-STABILITY IN BANACH SPACES

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Let $X$ be a real or complex Banach space. The norm on $X$ and on space $\mathcal{B}(X)$ of all bounded linear operators from $X$ into itself will be denoted by $\|*\|$. Let $T$ be the set of all pairs $(t, t_0)$ of positive real numbers satisfying the inequality $t \geq t_0$.

DEFINITION 1. *An evolution operator (on X) is a mapping* $\Phi : T \to \mathcal{B}(X)$ *with the properties:*

e1) $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$ *for all* $(s, t_0) \in T$.

e2) $\Phi(t, t) = I$ *(the identity operator on* $X$*) for every* $t \geq 0$.

e3) *For each* $t_0 \geq 0$ *and* $x_0 \in X$ *the function*

$$t \to \Phi(t, t_0)x_0$$

*is continuous on* $[t_0, \infty)$.

*Furthermore, if*

e4) *there are* $M \geq 1$ *and* $\omega > 0$ *such that*

$$\|\Phi(t, t_0)\| \leq M \cdot \exp[\omega(t - t_0)] \text{ for all } (t, t_0) \in T,$$

*then* $\Phi$ *is called an evolution operator with exponential growth.*

Uniform and nonuniform stability concepts are introduced by

DEFINITION 2. *An evolution operator* $\Phi : T \to \mathcal{B}(X)$ *is said to be:*

i) *stable (and we write s.) iff there exists* $M : \mathbf{R}_+ \to \mathbf{R}_+^*$ *such that*

$$\|\Phi(t, t_0)\| \leq M(t_0) \text{ for all } (t, t_0) \in T;$$

ii) *uniformly stable (and we write u.s.) iff there is* $M > 0$ *such that*

$$\|\Phi(t, t_0)\| \leq M \text{ for all } (t, t_0) \in T;$$

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iii) *exponentially stable (and we write e.s.) iff there are a, b, c > 0 such that*

$$\|\Phi(t, t_0)\| \le c \cdot \exp(at_0) \cdot \exp(-bt) \text{ for all } (t, t_0) \in T;$$

iv) *uniformly exponentially stable (and we write u.e.s.) iff there are* $N, \nu > 0$ *such that*

$$\|\Phi(t, t_0)\| \le N \cdot \exp[-\nu(t - t_0)] \text{ for all } (t, t_0) \in T.$$

Let $\mathcal{H}$ be the set of all nondecreasing functions $h: \mathbf{R}_+ \to \mathbf{R}_+^*$ and let $h \in \mathcal{H}$. If we denote

$$A_h = \left\{\omega \in \mathbf{R} : \sup_{t \ge 0} h(t) \cdot \exp(-\omega t) < \infty\right\}$$

and, respectively,

$$B_h = \left\{\omega \in \mathbf{R} : \inf_{t \ge 0} h(t) \cdot \exp(-\omega t) > 0\right\},$$

then the numbers

$$\overline{\omega}_h \overset{d}{=} \begin{cases} \inf A_h & ; A_h \ne \emptyset \\ \infty & ; A_h = \emptyset \end{cases} \quad \text{and} \quad \underline{\omega}_h \overset{d}{=} \begin{cases} \sup B_h & ; B_h \ne \emptyset \\ -\infty & ; B_h = \emptyset \end{cases}$$

are called *the upper* respectively *the lower characteristic number of h*.

PROPOSITION 1. *The evolution operator* $\Phi$ *is:*

i) *with exponential growth iff there is* $\varphi \in \mathcal{H}$ *such that* $\overline{\omega}_\varphi < \infty$ *and*

$$\|\Phi(t, t_0)\| \le \varphi(t - t_0) \text{ for all } (t, t_0) \in T;$$

ii) *exponentially stable iff there are* $h, k \in \mathcal{H}$ *with* $\underline{\omega}_h > 0$ *and* $\overline{\omega}_k < \infty$ *such that*

$$h(t) \cdot \|\Phi(t, t_0)\| \le k(t_0) \text{ for all } (t, t_0) \in T;$$

iii) *uniformly exponentially stable iff there is* $h \in \mathcal{H}$ *with* $\overline{\omega}_h < 0$ *such that*

$$\|\Phi(t, t_0)\| \le h(t - t_0) \text{ for all } (t, t_0) \in T.$$

*Proof.* It follows from:

i) $\overline{\omega}_h < \infty$ iff there are $M, \omega > 0$ with $h(t) \le M \cdot \exp(\omega t)$;

ii) $\underline{\omega}_h > 0$ iff there exist $m, b > 0$ with $h(t) \ge m \cdot \exp(bt)$;

iii) $\overline{\omega}_h < 0$ iff there are $N, \nu > 0$ such that $h(t) \le N \cdot \exp(-\nu t)$. □



DEFINITION 3. *Let* $h, k \in \mathcal{H}$. *The evolution operator* $\Phi: T \to \mathcal{B}(X)$ *is called (h, k)-stable (and we write (h, k)-s.) iff there exists* $c > 0$ *such that*

$$h(t) \cdot \|\Phi(t, t_0)x_0\| \le c \cdot k(t_0) \cdot \|x_0\| \text{ for all } (t, t_0) \in T \text{ and } x_0 \in X.$$

Let $\mathcal{E}$ be the set of all functions $h: \mathbf{R}_+ \to \mathbf{R}_+^*$ with the property that there is $\alpha > 0$ such that

$$h(t) = \exp(\alpha t) \text{ for every } t \in \mathbf{R}_+.$$

If we denote by $h_1$ the constant function $h_1(t) = 1$ for all $t \in \mathbf{R}_+$, then we have

PROPOSITION 2. *The evolution operator* $\Phi$ *is:*

i) *stable iff there is* $k \in \mathcal{H}$ *such that* $\Phi$ *is* $(h_1, k)$-s.;

ii) *uniformly stable iff it is* $(h_1, h_1)$-s.;

iii) *exponentially stable iff there are* $h, k \in \mathcal{E}$ *such that* $\Phi$ *is (h, k)-s.;*

iv) *uniformly exponentially stable iff there are* $h, k \in \mathcal{E}$ *with* $h \ge k$ *such that* $\Phi$ *is (h, k)-s.*

*Proof.* It is obvious from Definition 2 and Definition 3. □

A necessary condition for (h, k)-stability is given by

THEOREM 1. *If* $\Phi: T \to \mathcal{B}(X)$ *is (h, k)-stable, then for every* $H \in \mathcal{H}$ *with* $\overline{\omega}_{\frac{H}{h}} < 0$ *there is* $c > 0$ *such that*

$$\int_{t_0}^{\infty} H(t) \cdot \|\Phi(t, t_0)x_0\| dt \le c \cdot k(t_0) \cdot \|x_0\| \tag{1}$$

*for all* $t_0 > 0$ *and* $x_0 \in X$.

*Proof.* If $\Phi$ is (h, k)-s., then there exist $N, c, \nu > 0$ such that

$$\int_{t_0}^{\infty} H(t) \cdot \|\Phi(t, t_0)x_0\| dt \le N \int_{t_0}^{\infty} \exp(-\nu t) \cdot h(t) \cdot \|\Phi(t, t_0)x_0\| dt \le$$

$$\le N \cdot c \cdot k(t_0) \cdot \|x_0\| \int_{t_0}^{\infty} \exp(-\nu t) dt \le N \cdot c \cdot k(t_0) \cdot \|x_0\|$$

for all $t_0 > 0$ and $x_0 \in X$. □

A sufficient condition for (h, k)-stability is

THEOREM 2. *Let* $\Phi: T \to \mathcal{B}(X)$ *be an evolution operator with exponential growth. If there exist* $c > 0$ *and* $H \in \mathcal{H}$ *such that* (1) *and* $h(t + t_0) \le H(t) \cdot H(t_0)$ *hold for all* $t, t_0 \ge 0$ *and* $x_0 \in X$, *then* $\Phi$ *is (h, k)-s.* □

*Proof.* If $t_0 \geq 0$ and $t \geq t_0 + 1$, then

$$h(t) \cdot \|\Phi(t, t_0)x_0\| \cdot \int_0^1 \frac{du}{\varphi(u)H(u)} \leq \int_{t-1}^{t} \frac{h(t) \cdot \|\Phi(t, s)\| \cdot \|\Phi(s, t_0)x_0\|}{\varphi(t-s) \cdot H(t-s)} ds \leq$$

$$\leq \int_{t-1}^{t} H(s) \cdot \|\Phi(s, t_0)x_0\| ds \leq \int_{t_0}^{\infty} H(s) \cdot \|\Phi(s, t_0)x_0\| ds \leq c \cdot k(t_0) \cdot \|x_0\|$$

and hence

$$h(t) \cdot \|\Phi(t, t_0)x_0\| \leq c_1 \cdot k(t_0) \cdot \|x_0\|$$

for all $t \geq t_0 + 1$, $t_0 \geq 0$ and $x_0 \in X$, where

$$c_1 = \frac{c}{\int_0^1 \frac{du}{\varphi(u)h(u)}}.$$

It follows that $\Phi$ is $(h, k)$-s. □

As a particular case, we obtain a characterization of the exponential stability property given by

COROLLARY 1. *An evolution operator $\Phi : T \to \mathcal{B}(X)$ is exponentially stable if and only if there are a, b, $c > 0$ such that*

$$\int_{t_0}^{\infty} \exp(bt) \cdot \|\Phi(t, t_0)x_0\| dt \leq c \cdot \exp(at_0) \cdot \|x_0\| \tag{2}$$

*for all $t_0 \geq 0$ and $x_0 \in X$.*

*Proof.* Necessity: If $\Phi$ is e.s., then there are $a$, $b > 0$ such that $\Phi$ is $(h, k)$-s., where

$$h(t) = \exp(2bt) \text{ and } k(t) = \exp(at) \text{ for all } t \geq 0.$$

Then for $H(t) = \exp(bt)$ it follows that $\overline{\omega}_{\frac{H}{h}} < 0$ and the inequality (2) is satisfied.

Sufficiency: It is obvious that from Theorem 2 for

$$H(t) = h(t) = \exp(bt) \text{ and } k(t) = \exp(at)$$

it follows that $\Phi$ is e.s. □



A characterization of the uniform exponential property is given by

COROLLARY 2. *An evolution operator $\Phi : T \to \mathcal{B}(X)$ is uniformly exponentially stable if and only if there exist a, $b > 0$ with $b \geq a$ such that the inequality (2) is satisfied.*

*Proof.* It results similarly to the proof of Corollary 1. □

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