REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

Tome XXVI, N^{os} 1–2, 1997, pp. 59–74

NUMERICAL STABILITY OF COLLOCATION METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

In [3] we have presented a method for the construction of an approximation to the solution of the following initial-value problem for the first-order Volterra integro-differential equation (VIDE)

$$(1.1)\qquad y'(t) = f(t, y(t)) + \int_0^t K(t, s, y(s))\,\mathrm{d}s,\ t \in I := [0,T],$$

with the initial condition $y(0) = y_0$, by polynomial spline functions. Here, the given functions $f: I \times R \to R$ and $K: S \times R \to R$ (with $S := \{(t,s): 0 \le s \le t \le T\}$) are supposed to be sufficiently smooth for the initial-value problem for VIDE (1.1) to have a unique solution $y \in C^{\alpha}(I)$, with $\alpha \in \mathbf{N}$ (see [6]).

In order to describe this method, let $\Pi_N: 0 = t_0 < t_1 < \ldots < t_N = T$ (with $t_n = t_n^{(N)}$) be a quasi-uniform mesh for the given interval I, and set

$$\sigma_0 := [t_0, t_1],\ \sigma_n := (t_n, t_{n+1}],\ \text{for } n = 1, 2, \ldots, N-1,$$
$$h_n := t_{n+1} - t_n,\ \text{for } n = 0, 1, \ldots, N-1,$$
$$h = \max\{h_n : n = 0, 1, \ldots, N-1\},$$
$$Z_N := \{t_n : n = 1, \ldots, N-1\},\ \overline{Z_N} = Z_N \cup \{T\}.$$

Moreover, let $\mathcal{P}_k$ denote the space of (real) polynomials of a degree not exceeding k. Then we define, for given integers m and d with $m \ge 1$ and $d \ge -1$,

$$S_{m+d}^{(d)}(Z_N) := \{u : u(t)\big|_{t \in \sigma_n} =: u_n(t) \in \mathcal{P}_{m+d},\ n = 0, \ldots, N-1,$$
$$u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n)\ \text{for } j = 0, 1, \ldots, d \text{ and } t_n \in Z_N\},$$

1991 AMS Subject Classification: 65R20.

to be the space of polynomial splines of degree $m+d$, whose elements possess the knots Z_N and are d-times continually differentiable on I. If $d=-1$, then the elements of $S^{(-1)}_{m-1}(Z_N)$ may have jump discontinuities at the knots Z_N.

An element $u \in S^{(d)}_{m+d}(Z_N)$ has for all $n = 0, 1, \ldots, N-1$ and for all $t \in \sigma_n$ the following form (see [7])

$$u(t) = u_n(t) = \sum_{r=0}^{d} \frac{u^{(r)}_{n-1}(t_n)}{r!}(t - t_n)^r + \sum_{r=1}^{m} a_{n,r}(t - t_n)^{d+r}. \tag{1.2}$$

From (1.2) we see that an element $u \in S^{(d)}_{m+d}(Z_N)$ is well defined when we know the coefficients $\{a_{n,r}\}_{r=\overline{1,m}}$ for all $n = 0,\ldots, N-1$. In order to determine these coefficients, we consider the set of collocation parameters $\{c_j\}_{j=\overline{1,m}}$, where $0 < c_1 < \ldots < c_m \le 1$, and we define the set of collocation points by

$$X(N) := \bigcup_{n=0}^{N-1} X_n, \text{ with } X_n := \{t_{n,j} := t_n + c_j h_n,\ j = 1, 2, \ldots, m\}.$$

The approximate solution $u \in S^{(d)}_{m+d}(Z_N)$ will be determined by imposing the condition that u satisfies the initial-value problem (1.1) on $X(N)$

$$u'(t) = f(t, u(t)) + \int_0^t K(t, s, u(s))\,ds,\ t \in X(N), \text{ with } u(0) := y_0. \tag{1.3}$$

The above algorithm determines a unique approximate solution $u \in S^{(d)}_{m+d}(Z_N)$ whose convergence and local superconvergence properties have been studied in [3].

In this paper, we will analyze the numerical stability of the polynomial spline collocation method in the case in which the mesh sequences $\{\Pi_N\}_N$ are uniform, i.e., $h_n = h$, for all $n = 0, 1,\ldots, N-1$.

2. NUMERICAL STABILITY

In order to discuss numerical stability, we study the behavior of the method as applied to the Volterra integro-differential equation

$$y'(t) = f(t) + \alpha y(t) + \lambda \int_0^t y(s)\,ds,\ \lambda \ne 0, \tag{2.1}$$

with the initial condition $y(0) = y_0$. Here, the given function $f : I \to R$ is supposed to be sufficiently smooth (i.e., $f \in C^{\alpha}(I)$, with $\alpha \ge 1$).

This equation is called the basis test equation and it was suggested by Brunner and Lambert in 1974 (see [2]), and then it has been extensively used for investigating stability properties of several methods.

Henceforward, we will refer to a polynomial spline collocation method in the space $S^{(d)}_{m+d}(Z_N)$, simply as an *(m, d)-method* (see [4], [5]).

DEFINITION 2.1. *An (m, d)-method is said to be stable if all solutions* $\{u(t_n)\}$ *remain bounded, as* $n \to \infty, h \to 0$ *while hN remains fixed.*

From relation (1.2) we observe that the first $d+1$ coefficients of the polynomial $u \in S^{(d)}_{m+d}(Z_N)$ are determined by the smooth condition, and the last m coefficients are determined by the collocation conditions. Thus, it is convenient to introduce the following notations:

$$\eta_n := (\eta_{n,r})_{r=\overline{0,d}}, \text{ with } \eta_{n,r} := \frac{u^{(r)}_{n-1}(t_n)}{r!} h^r, \text{ and}$$
$$\beta_n := (\beta_{n,r})_{r=\overline{1,m}}, \text{ with } \beta_{n,r} := a_{n,r} h^{d+r}, (n = 0, 1, \ldots, N). \tag{2.2}$$

With these notations, for all $t := t_n + \tau h \in \sigma_n$, (1.2) becomes

$$u(t) = u_n(t_n + \tau h) = \sum_{r=0}^{d} \eta_{n,r}\tau^r + \sum_{r=1}^{m} \beta_{n,r}\tau^{d+r}, \quad \text{for all } \tau \in (0, 1] \text{ and } n = 0, 1, \ldots, N. \tag{2.3}$$

Now, for $d \ge 1$, if we apply the collocation method to test integral equation (2.1) and we use the representation (2.3), we obtain the following collocation equation

$$V\beta_n = W\eta_n + hr_n, \text{ for all } n = 0, 1, \ldots, N-1, \tag{2.4}$$

where V is the $m \times m$ matrix, W is the $m \times (d+1)$ matrix, and r_n is the m-vector, whose elements are

$$v_{j,r} := \left((d+r) - \alpha h c_j - \frac{\lambda h^2 c_j}{d+r+1}\right) c_j^{d+r-1};$$

$$w_{j,r} := \begin{cases} \lambda h^2 c_j, & \text{if } r = 0, \\ \left(\alpha h + \dfrac{\lambda h^2 c_j}{2}\right) c_j, & \text{if } r = 1, \\ \left(\alpha h c_j + \dfrac{\lambda h^2 c_j^2}{r+1} - r\right) c_j^{r-1}, & \text{if } r \ge 2; \end{cases}$$

and

$$r_{n,j} := \begin{cases} f(t_{0,j}) - f(t_0), \text{ if } n = 0, \\ f(t_{n,j}) - f(t_{n-1,m}) + u'_{n-1}(t_{n-1,m}) - u'_{n-1}(t_n) + \alpha\left[u_{n-1}(t_n) - u_{n-1}(t_{n-1,m})\right] + \\ +\lambda h \int_{c_m}^{1} u_{n-1}(t_{n-1} + \tau h), \text{ if } n > 0. \end{cases}$$

By direct differentiation of relations (2.3), for the smooth conditions of the approximation $u \in S^{(d)}_{m+d}(Z_N)$, we get a relation between vector η_{n+1} and vectors η_n and β_n, respectively

$$\eta_{n+1} = A\eta_n + B\beta_n, \text{ for all } n = 0, 1, \ldots, N-2, \tag{2.5}$$

where A is the $(d+1)\times(d+1)$ upper triangular matrix, and B is the $(d+1)\times m$ matrix, whose elements are

$$a_{j,r} := \begin{cases} 0, & \text{if } r < j \\ \binom{r}{j}, & \text{if } r \ge j \end{cases}, \quad b_{j,r} := \binom{d+r}{j}.$$

In order to prove the results concerning the numerical stability properties of the polynomial spline collocation method, we need the following lemma (see [8]):

LEMMA 2.2. *For any matrix P and any $\varepsilon > 0$, there exists a subordinate norm such that $\|P\| \le S(P) + \varepsilon$, with $S(P) := \max\{|\lambda_j| : \lambda_j$ are the eigenvalues of $P\}$. If P is of class **M**, then there exists a norm such that $\|P\| = S(P)$.*

By means of this lemma we can characterize numerical stability in the terms of eigenvalues of the suitable matrix. The following theorem represents a stability criterion for our method:

THEOREM 2.3. *An (m, d)-method is stable if and only if all eigenvalues of matrix $M := A + BV^{-1}W$ are in the unit disk and all eigenvalues with $|\mu| = 1$ belong to 1×1 Jordan block.*

Proof. In order to prove this theorem, we will show that the vectors η_n and β_n, defined by (2.2), are uniformly bounded for $h \searrow 0, n \to \infty$, while hN remains fixed, i.e., there exist two finite constants M_1 and M_2, such that

$$\|\beta_n\|_1 := \sum_{j=1}^{m} |\beta_{n,j}| \le M_1, \text{ and } \|\eta_n\|_1 := \sum_{j=1}^{m} |\eta_{n,j}| \le M_2,$$

uniformly in n, as $h \searrow 0$. These in turn imply, according to (2.3), that

$$|u(t_n)| \le M_1 + M_2, \text{ for all } n = 0, 1, \ldots, N-1,$$

and from Definition 2.1, it results that an *(m, d)-method* is stable.

From the form of matrix V we see that for h small enough, this matrix is nonsingular. Elimination of β_n between (2.4) and (2.5) yields

$$\eta_{n+1} = M\eta_n + BV^{-1}r_n, \text{ with } M := A + BV^{-1}W, \tag{2.6}$$

for all $n = 0, 1, \ldots, N-2$. Thus, relations (2.4) and (2.6) imply that for all $n = 0, 1, \ldots, N-1$, we have

$$\eta_n = M^n\eta_0 + \sum_{i=0}^{n-1} M^{n-1-i}BV^{-1}r_i, \qquad \beta_n = V^{-1}W\left[M^n\eta_0 + \sum_{i=0}^{n-1} M^{n-1-i}BV^{-1}r_i\right] + V^{-1}r_n. \tag{2.7}$$

Because the first derivative of the given function f is a continuous function on I, it results that there exists a positive constant L such that $|f'(t)| \le L$, for all $t \in I$; and, for all $n = 0, \ldots, N-1$, we have

$$\|r_n\|_1 := \sum_{j=1}^{m} |r_{n,j}| \le \sum_{j=1}^{m} \Big[Lh\left(1 - c_m + c_j\right) + |u'_{n-1}(t_{n-1,m}) - u'_{n-1}(t_n)| + \\ +\alpha|u_{n-1}(t_{n-1,m}) - u_{n-1}(t_n)| + |\lambda| h \int_{c_m}^{1} |u_{n-1}(t_{n-1} + \tau h)| \,d\tau \Big]. \tag{2.8}$$

In the case in which $c_m = 1$, relation (2.8) becomes

$$\|r_n\|_1 \le hL \sum_{j=1}^{m} c_j \le hL_1, \text{ with } L_1 := L\sum_{j=1}^{m} c_j \le mL, \tag{2.9}$$

and from relation (2.7) we obtain

$$\|\eta_n\|_1 \le \|M\|^n \|\eta_0\|_1 + hL_1 \|BV^{-1}\|_1 \sum_{i=0}^{n-1} \|M\|_1^{n-1-i}, \qquad \|\beta_n\|_1 \le \|V^{-1}W\|_1 \cdot \|\eta_n\|_1 + h\|V^{-1}\|_1 L_1, \; n = 0, 1, \ldots, N-1. \tag{2.10}$$

Using Lemma 2.2, it results from (2.10) that

$$
\begin{aligned}
&\|\eta_n\|_1 \le L(S(M))^n\|\eta_0\|_1 + hL_1\|BV^{-1}\|_1\sum_{i=0}^{n-1}(S(M))^{n-1-i}, \\
(2.11)\qquad &\|\beta_n\|_1 \le \|V^{-1}W\|_1 \|\eta_n\|_1 + h\|V^{-1}\|_1 L_1, \quad (n = 0,\ldots, N-1).
\end{aligned}
$$

From these relations, it results that $\|\eta_n\|_1$ and $\|\beta_n\|_1$ remain bounded for $n \to \infty, h \to 0$ and $Nh = T$, if and only if $S(M) \le 1$.

In the case in which $c_m \ne 1$, then we will prove by induction that relations (2.9) and (2.10) hold if we change the constant L_1, in (2.9), with a new finite constant $L_{2,n}$, defined by

$$
L_{2,n} := \begin{cases} L_1, & \text{if } n = 0, \\ L\sum_{j=1}^{m}(1 - c_m + c_j) + m(1 - c_m)\left(M_n^{(2)} + \alpha M_n^{(1)} + |\lambda| M_n^{(0)}\right), & \text{if } n \ge 1, \end{cases}
$$

where

$$
M_n^{(i)} := \begin{cases} 0, & \text{if } n = 0, \\ \max\left\{\left|u_{n-1}^{(i)}(t)\right| : t \in \sigma_{n-1}\right\}, & \text{if } n \ge 1, \end{cases} \qquad \text{for } i = 0,1,2,
$$

and, respectively, in (2.10) we take

$$
(2.12)\qquad L_2 := \max\{L_{2,n} : n = 0,1,\ldots, N-1\}.
$$

For $n = 0$, relations (2.7) become: $\eta_0 = \eta_0$ and $\beta_0 = V^{-1}W\eta_0 + V^{-1}r_0$, respectively. Because the matrices V^{-1}, W and the vector r_0 are bounded in norm for $h \to 0$, it results that the vector β_0 is bounded, too. Thus, by the definition relation (2.3), we obtain $|u_0(\tau h)| < \infty$, and $|u'_0(\tau h)| < \infty$, for all $\tau \in [0,1]$, hence, by (2.8), it results that $\|r_1\|_1 \le hL_{2,1}$, with $L_{2,1} < \infty$.

Now we suppose that $\|\beta_j\|_1 \le \infty$ and $\|\eta_j\|_1 < \infty$, for all $j = 0, 1,\ldots, n-1$. Under this assumption, by (2.3) it results that $|u_{n-1}(t)| < \infty$, and $|u'_{n-1}(t)| < \infty$, for all $t \in \sigma_{n-1}$; furthermore, by (2.8) it follows that $\|r_n\|_1 \le hL_{2,n}$, with $L_{2,n} < \infty$. Moreover, relations (2.5) and (2.4) imply $\|\eta_n\|_1 < \infty$, and $\|\beta_n\|_1 < \infty$, respectively. Thus, using the bound of r_n, from (2.7) it results that relations (2.10) and (2.11) hold with L_1 replaced by L_2, for all $n = 0, 1,\ldots, N-1$; accordingly, the theorem is fully demonstrated.

Remark 2.4. From (2.6) we see that the dimension of matrix M is $\dim M := d+1$. *Moreover, if we denote by M_0 the matrix M with $h = 0$, and by $\mu^{(0)}$ and μ the eigenvalues of M_0 and M, respectively, then it follows that the matrix M_0 has $\mu_1^{(0)} = \mu_2^{(0)} = 1$, for all $m \ge 0$ and $d \ge 1$.*

3. APPLICATIONS

In the following we will investigate some special cases.

I. $d = 1$. In this case the approximation space is $S_{m+1}^{(1)}(Z_N)$. From Theorem 2.3 and Remark 2.4, the following theorem results:

THEOREM 3.1. *An (m, 0)-method is stable for all $m \ge 1$, and for every choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$.*

The above theorem may be directly proved by using the same technique as in the first application from [4].

II. $m = 1$. This choice of m corresponds to a classical spline function, i.e., $u \in S_{d+1}^{(d)}(Z_N)$, $d \ge 1$. Using notations from Remark 2.4 (i.e., M_0 is the matrix M, with $h = 0$, and by $\mu^{(0)}$ and μ, the eigenvalues of M_0 and M, respectively), we have

$$
\mu = \mu^{(0)} + O(h).
$$

If $c_1 \in (0,1]$ is the collocation parameter, then, for all $d \ge 1$, using the binomial expansion, we find for matrix M_0 the trace

$$
(3.1)\qquad Tr(M_0) = d + 2 + \frac{1}{c_1^d} - \left(1 + \frac{1}{c_1}\right)^d.
$$

As regards the stability of the spline collocation method, we have the following result:

THEOREM 3.2. *A (1, d)-method is stable if and only if one from the following conditions is true:*

(i) *$d = 1$ and $c_1 \in (0,1]$;*
(ii) *$d = 2$ and $c_1 = 1$.*

Proof. In the case $d = 1$, this theorem follows from Theorem 3.1. If $d = 2$, then the third eigenvalue of M_0 is $\mu_3^{(0)} = 1 - \dfrac{2}{c_1} \le -1$, for $c_1 \in (0,1]$, and its absolute value is 1, if and only if $c_1 = 1$. For $d \ge 3$, from relation (3.1), we obtain

$$-\infty < Tr(M_0) < -(d+1), \text{ if } d > 3 \text{ and } c_1 \in (0,1],$$

and $\mu_2^{(0)} + \mu_3^{(0)} \le -4$, if $d = 2$. Since $Tr(M_0) = \mu_1^{(0)} + \mu_2^{(0)} + \ldots + \mu_{d+1}^{(0)} < -(d+1)$, it results that there exists an eigenvalue $\mu^{(0)}$ whose value is smaller than -1, i.e., $|\mu^{(0)}| > 1$. Thus, from Theorem 2.3 we have that, for $d \ge 2$, a *(1, d)-method* is unstable for every choice of the collocation parameter $c_1 \in (0,1]$.

III. $m = 2$. In this case, we find for the trace of matrix M_0 the relation

$$(3.2)\qquad Tr(M_0) = d + 3 + \frac{(1+c_2)^d(c_1 - c_2 - 1) + (1 - c_1)}{c_2^d(c_2 - c_1)} - \frac{(1+c_1)^d(c_2 - c_1 - 1) + (1 - c_2)}{c_1^d(c_2 - c_1)},$$

where $0 < c_1 < c_2 \le 1$ are the collocation parameters. Using the above relation, we obtain the following

THEOREM 3.3. (i) *A (2, 1)-method is stable for every choice of the collocation parameters.*

(ii) *A (2, 2)-method is stable if and only if* $c_1 + c_2 \ge \frac{3}{2}$.

(iii) *If* $c_2 = 1$, *then a (2, d)-method is unstable for all* $d \ge 3$.

Proof. Assertion (i) follows from Theorem 3.1. To prove assertion (ii), it is enough to observe that, for $d = 1$, the third eigenvalue of M_0 is $\mu_3^{(0)} = \dfrac{c_1c_2 - 2(c_1 + c_3) + 3}{c_1c_2}$, and the stability condition $|\mu_2^0| \le 1$ is equivalent to the condition $c_1 + c_2 \ge \frac{3}{2}$.

If $d = 2$ and $c_2 = 1$, then one of the eigenvalues of M_0 is

$$\mu_3^{(0)} = \frac{1}{2c_1^2}\left(-4c_1^2 + 4c_1 + 1 + \sqrt{12c_1^4 - 24c_1^3 + 4c_1^2 + 8c_1 + 1}\right);$$

here we have $\mu_3^{(0)} > 1$ for every choice of the collocation parameter $c_1 \in (0,1)$. Thus, assertion (iii) holds for $d = 3$. If $d > 3$, then for $c_2 = 1$ the formula (3.2) becomes

$$Tr(M_0) = d + 4 + \frac{c_1\left[2^d + \left(1 + \frac{1}{c_1}\right)^d\right] - 2^{d+1}}{(1 - c_1)} \ge d + 4,$$

for all $c_1 \in (0,1)$, and thus the assertion of this theorem follows from Theorem 2.3.

IV. $d = 2$. In this case, approximation $u \in S_{m+2}^{(2)}(Z_N)$, the dimension of the matrix M_0 is 3, and $\mu_1^{(0)} = \mu_2^{(0)} = 1$ are its first two eigenvalues. By direct computation, for the third eigenvalue of M_0, we find

$$(3.3)\qquad \mu_3^{(0)} = \frac{S_m - 2S_{m-1} + 3S_{m-2} + \ldots + (-1)^{m-1} mS_1 + (-1)^m (m+1)}{S_m},$$

if $m = 1, 2, 3, 4, 5, 6,$

where

$$(3.4)\qquad S_K := \sum_{1 \le i_1 < \ldots < i_k \le m}^{m} c_{i_1} c_{i_2} \ldots c_{i_k}, \text{ for } 1 \le k \le m.$$

In view of the results obtained for $m = 1, 2, \ldots, 6$ we are led to the following affirmation:

Conjecture 3.4. *If* $d = 2$, *then the third eigenvalue of* M_0 *may be calculated by using relation* (3.3) *for all* $m \ge 1$.

Now, if we denote by $R_m(t)$ the polynomial of degree m whose zeros are the collocation parameters $\{c_j\}_{j=\overline{1,m}}$, then we have the following stability criterion:

THEOREM 3.5. *An (m, 2)-method is stable if and only if*

$$(3.5)\qquad \left| \frac{\left[\frac{\mathrm{d}}{\mathrm{d}t}(t \cdot R_m(t))\right]_{t=1}}{R_m(0)} \right| \le 1.$$

Proof. Since $R_m(t)$ is the polynomial of degree m whose zeros are the collocation parameters $\{c_j\}_{j=\overline{1,m}}$, using notation (3.4), it may be written

$$(3.6)\qquad R_m(t) = t^m - S_1 t^{m-1} + S_2 t^{m-2} + \ldots + (-1)^m S_m.$$

Thus, from (3.3), (3.5) and (3.6), we obtain

$$\mu_3^{(0)} = \frac{\left[\frac{\mathrm{d}}{\mathrm{d}t}(t \cdot R_m(t))\right]_{t=1}}{R_m(0)};$$

and so, if Conjecture 3.4 is true, the assertion of this theorem follows from Theorem 2.3.

COROLLARY 3.6. *If the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are uniformly distributed in* (0, 1] $\left(i.e., c_j := \frac{j}{m}, \text{for all } j = 1, 2, \ldots, m\right)$, *then an (m, 2)-method is stable.*

If Conjecture 3.4 is true, then for $c_m = 1$ the above conjecture and theorem become

COROLLARY 3.7. *If the last collocation parameter is one (i.e.,* $c_m = 1$*), then:*

(i) *The third eigenvalue of* M_0 *may be calculated by using the relation*

$$\mu_3^{(0)'} = (-1)^m \frac{1 - S'_1 + S'_2 - S'_3 + \ldots + (-1)^{m-1} S'_{m-1}}{S_{m-1}}, \tag{3.7}$$

for all $m \geq 1$, *where*

$$S'_k = \sum_{1 \leq i_1 < \ldots < i_k \leq m-1} c_{i_1} c_{i_2} \ldots c_{i_k}, \text{ for } 1 \leq k \leq m-1. \tag{3.8}$$

(ii) *An (m, 2)-method is stable if and only if*

$$\left| \frac{\left[\frac{\mathrm{d}}{\mathrm{d}t}(R_m(t))\right]_{t=1}}{R_m(0)} \right| \leq 1, \tag{3.9}$$

where $R_m(t)$ *is the polynomial of degree m defined by* (3.8).

Proof. (i) If the last collocation parameter is one (i.e., $c_m = 1$), then, from (3.4) it results that

$$S_k = \begin{cases} S'_1 + 1, & \text{if } k = 1; \\ S'_k + S'_{k-1}, & \text{if } 2 \leq k \leq m-1, \\ S'_{m-1}, & \text{if } k = m \end{cases} \tag{3.10}$$

where S_k' are defined in (3.8). Now, the first assertion of this corollary follows by Conjecture 3.4 and relation (3.10).

(ii) Using notations (3.8), the polynomial R_m, whose zeros are the collocation parameters $\{c_j\}_{j=\overline{1,m}}$, may be written

$$R_m(t) = (t-1)\left(t^{m-1} - S'_1 t^{m-2} + S'_2 t^{m-3} + \ldots + (-1)^{m-1} S'_{m-1}\right), \text{ for all } t \in [0,1]. \tag{3.11}$$

Thus, from (3.7), (3.9) and (3.11), we obtain

$$\left|\mu_2^{(0)'}\right| = \left| \frac{\left[\frac{\mathrm{d}}{\mathrm{d}t}(R_m(t))\right]_{t=1}}{R_m(0)} \right|;$$

and so, the second assertion of this corollary follows from Theorem 2.3.

In [3] we have proved that, in a suitable choice of the collocation parameters, we obtain an approximated solution which has a local convergence order greater than the global order, in the points from Z_N. As regards the stability of this local superconvergent solution $u \in S_{m+2}^{(2)}(Z_N)$, we have

COROLLARY 3.8. (i) *If the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are the Radau II points from* (0, 1], *then an (m, 2)-method is unstable for all* $m \geq 2$.

(ii) *If the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are the Gauss points from* (0, 1), *then an (m, 2)-method is unstable for all* $m \geq 2$.

(iii) *If the first* $m - 1$ *collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are the Gauss points from* (0, 1), *and the last is* $c_m = 1$, *then an (m, 2)-method is stable for all* $m \geq 2$.

Proof. The results from this corollary follow from assertion (ii) of Corollary 3.7 and the properties of the Radau II points and Gauss points, respectively. In this proof we will denote by $P_m(s)$ the Legendre's polynomial of a degree not expanding m, for $s \in [-1, 1]$.

(i) If the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ are the Radau II points from (0, 1], then the polynomial R_m, whose zeros are the collocation parameters $\{c_j\}_{j=\overline{1,m}}$, may be written

$$R_m(t) = P_{m-1}(2t-1) - P_m(2t-1), \text{ for all } t \in [0,1].$$

Thus, using the properties of Legendre's polynomial, from (3.9), we obtain

$$\left|\mu_2^{(0)'}\right| = 2\left| \frac{P'_{m-1}(1) - P'_m(1)}{P_{m-1}(-1) - P_m(-1)} \right| = m > 1, \text{ for all } m \geq 2.$$

(ii) If the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ are the Gauss points from (0, 1), then the polynomial R_m is

$$R_m(t) = P_m(2t-1), \text{ for all } t \in [0,1].$$

Because $P'_m(1) = \dfrac{m(m+1)}{2}$, from (3.9) it results that $\left|\mu_2^{(0)'}\right| = m(m+1) > 1$, for all $m \geq 2$.

(iii) In this choice of collocation parameters, polynomial R_m becomes

$$R_m(t) = (t-1)\cdot P_{m-1}(2t-1), \text{ for all } t \in [0,1];$$

and, from (3.9), we obtain

$$\left|\mu_2^{(0)'}\right| = \left|\frac{\left[\frac{d}{dt}(R_m(t))\right]_{t=1}}{R_m(c)}\right| = \left|\frac{P_{m-1}(1)}{P_{m-1}(-1)}\right| = 1,$$

for all $m \geq 1$.

V. $d = 0$. In the end of this section we analyze the numerical stability of the spline collocation method in the space $S_m^{(0)}$, for $m \geq 1$. An element $u \in S_m^{(0)}(Z_N)$ has for all $n = 0, 1, \ldots, N-1$ the form

$$\text{(3.12)} \qquad u_n(t_n + \tau h) = u_{n-1}(t_n) + \sum_{r=1}^{m} \beta_{n,r}\tau^r, \text{ for } \tau \in (0,1].$$

If we denote by u_{n+1} and by u'_{n+1} the vectors with m-elements

$$u_{n+1} := \left(u_n\left(t_n + c_j h\right)\right)^T_{j=\overline{1,m}}, \text{ and } u'_{n+1} := \left(u'_n\left(t_n + c_j h\right)\right)^T_{j=\overline{1,m}},$$

then from equation (3.12) we obtain

$$\text{(3.13)} \qquad u_{n+1} = (1,1,\ldots,1)^T u_{n-1}(t_n) + E\cdot\beta_n, \text{ for } n = 0,1,\ldots,N-1,$$

$$\text{(3.14)} \qquad u'_{n+1} = h^{-1}E'\cdot\beta_n, \text{ for } n = 0,1,\ldots,N-1,$$

with the matrices E and E' defined by $E := \left(c_j^r\right)_{j,r=\overline{1,m}}$ and $E := \left(rc_j^{r-1}\right)_{j,r=\overline{1,m}}$, respectively.

In this case the collocation equation becomes

$$\text{(3.15)} \quad V\beta_n = hW_0\left(u_{n-1}(t_n), u'_{n-1}(t_n)\right)^T + r_n, \text{ for all } n = 0,1,\ldots,N-1,$$

where matrix W_0 is defined by

$$W_0 := \left(w^0_{j,r}\right)_{j=\overline{1,m}, r=1,2}, \text{ with } w^0_{j,r} := \begin{cases} \lambda h c_j, & \text{if } r = 1; \\ 1, & \text{if } r = 2. \end{cases}$$

Here, matrix V and vector r_n are like in (2.4).

Because $V = E' + O(h)$, the elimination of β_n between (3.14) and (3.15) yields

$$\text{(3.16)} \quad \begin{array}{c} u'_n(t_{n,j}) = (1+O(h))u'_{n-1}(t_n) + (1+O(h))r_{n,j} + O(h)u_{n-1}(t_n), \\ \text{for all } j = 1,2,\ldots,m \ (n = 0,1,\ldots,N-1). \end{array}$$

For all $\tau \in [0,1]$, the first derivatives of the approximation $u \in S_m^{(0)}(Z_N)$ may be written

$$\text{(3.17)} \qquad u'_n(t_n + \tau h) = \sum_{j=1}^{m} L_j(\tau)u'_n(t_{n,j}), \text{ for all } n = 0,1,\ldots,N-1,$$

where

$$L_j(\tau) := \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{(\tau - c_i)}{(c_j - c_i)}, \text{ for all } j = 0,1,\ldots,m,$$

are the Lagrange fundamental polynomial associated with the collocation parameters $\{c_j\}_{j=\overline{1,m}}$. Now, replacing $u'_n(t_{n,j})$ in (3.17) with its values given by (3.16), for all $n = 0, 1, \ldots, N-1$, we obtain

$$\text{(3.18)} \quad u'_n(t_{n+1}) = hO(h)u_{n-1}(t_n) + (1+O(h))\left(u'_{n-1}(t_n) + \sum_{j=1}^{m} L_j(1)r_{n,j}\right),$$

for all $n = 0, 1, \ldots, N-1$.

By integrating relation (3.17), for $\tau \in [0,1]$, and using again relation (3.16), we obtain

$$\text{(3.19)} \quad \begin{aligned} u_n(t_{n+1}) &= (1 + hO(h))u_{n-1}(t_n) + h(1+O(h))u'_{n-1}(t_n) + \\ &+ h(1+O(h))\int_0^1 \sum_{j=1}^{m} L_j(\tau)r_{n,j}, \text{ for all } n = 0,1,\ldots,N-1. \end{aligned}$$

Equations (3.18) and (3.19) form together a system which may be written

$$\text{(3.20)} \quad \begin{pmatrix} u_n(t_{n+1}) \\ u'_n(t_{n+1}) \end{pmatrix} = M' \begin{pmatrix} u_{n-1}(t_n) \\ u'_{n-1}(t_n) \end{pmatrix} + (1+O(h))r'_n,$$

for all $n = 0,1,\ldots,N-1,$

where

$$M' := \begin{pmatrix} (1+hO(h)) & h(1+O(h)) \\ hO(h) & (1+O(h)) \end{pmatrix}, \; r'_n := \left(h\int_0^1 \sum_{j=1}^{m} L_j(\tau) r_{n,j}, \sum_{j=1}^{m} L_j(1) r_{n,j} \right)^T.$$

Equation (3.20) has the same form as equation (2.7). Thus, because for $h \to 0$ the matrix M' has the eigenvalues $\mu'_1 = \mu'_2 = 1$, as in proof of Theorem 2.3, we may prove the following

THEOREM 3.9. *An* $(m, 0)$*-method is stable for all* $m \geq 1$ *and for every choice of the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$.

4. A NUMERICAL EXAMPLE

We give below the results obtained when applying various $(3, d)$*-methods* to the following integro-differential equation of the first order

$$(4.1) \qquad \begin{aligned} y'(t) &= y(t) + 2t\exp(t^2) + \int_0^t 2t\exp(t^2 - s^2) y(s)\,\mathrm{d}s, \\ y(0) &= 1, \text{ for } t \in [0,1], \end{aligned}$$

whose exact solution is $y(t) = \exp(t + t^2)$.

In the following we use the notations: $e_1 := |y(t_1) - u(t_1)|$, $e_5 := |y(t_5) - u(t_5)|$, $e_N := |y(1) - u(1)|$, where $u \in S_3^{(d)}$ is the approximated solution and $t_i := ih \in Z_N$. Thus, for $N = 10$ ($h = 0.1$) we obtain:

a) If the collocation parameters are $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$ and $c_3 = 1$, then we have:

$$e_1 = 0.1\times10^{-6}, e_5 = 0.2\times10^{-5}, e_N = 0.3\times10^{-4} \text{ for } d = 1;$$

$$e_1 = 0.7\times10^{-8}, e_5 = 0.7\times10^{-7}, e_N = 0.3\times10^{-6} \text{ for } d = 2;$$

$$e_1 = 0.1\times10^{-8}, e_5 = 0.1\times10^{-4}, e_N = 3.350 \text{ for } d = 3.$$

b) If the collocation parameters are the *Radau II points*, i.e., $c_1 = \frac{4-\sqrt{6}}{10}$, $c_2 = \frac{4+\sqrt{6}}{10}$ and $c_3 = 1$, then we have:

$$e_1 = 0.2\times10^{-8}, e_5 = 0.5\times10^{-7}, e_N = 0.8\times10^{-6} \text{ for } d = 1;$$

$$e_1 = 0.6\times10^{-8}, e_5 = 0.7\times10^{-7}, e_N = 0.6\times10^{-5} \text{ for } d = 2;$$

$$e_1 = 0.1\times10^{-8}, e_5 = 0.2\times10^{-3}, e_N = 317390.7091 \text{ for } d = 3.$$

c) If the collocation parameters are the *Gauss points*, i.e., $c_1 = \frac{5-\sqrt{15}}{10}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{4+\sqrt{6}}{10}$, then we have:

$$e_1 = 0.1\times10^{-9}, e_5 = 0.3\times10^{-8}, e_N = 0.3\times10^{-7} \text{ for } d = 1;$$

$$e_1 = 0.1\times10^{-8}, e_5 = 0.4\times10^{-5}, e_N = 291.2755 \text{ for } d = 2;$$

$$e_1 = 0.1\times10^{-8}, e_5 = 0.0758, e_N = 0.433\times10^{13} \text{ for } d = 3.$$

d) If the first two collocation parameters are the *Gauss points*, i.e., $c_1 = \frac{3-\sqrt{3}}{10}$, $c_2 = \frac{3+\sqrt{3}}{6}$, and $c_3 = 1$, then we have:

$$e_1 = 0.2\times10^{-6}, e_5 = 0.3\times10^{-5}, e_N = 0.4\times10^{-4} \text{ for } d = 1;$$

$$e_1 = 0.9\times10^{-8}, e_5 = 0.4\times10^{-7}, e_N = 0.5\times10^{-6} \text{ for } d = 2;$$

$$e_1 = 0.1\times10^{-8}, e_5 = 0.3\times10^{-4}, e_N = 35.54725 \text{ for } d = 3.$$

From this numerical example we observe that a $(3, d)$*-method* is stable for $d = 1$ and it is unstable for $d = 3$. In the case $d = 2$, this method is stable if the collocation parameters are $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$, $c_3 = 1$ (i.e., case a)), or $c_1 = \frac{3-\sqrt{3}}{10}$, $c_2 = \frac{3+\sqrt{3}}{6}$, and $c_3 = 1$ (i.e., case d)).

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Received December 15, 1996

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