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CONDITIONS OF STABILITY, PSEUDO-STABILITY AND QUASI-STABILITY OF THE PARETO SET IN A VECTOR TRAJECTORIAL PROBLEM*

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It has been shown in [1–3] that the coincidence of Pareto set, Slater set and Smale set is a necessary and sufficient condition of stability of the Pareto set in vector problems of discrete optimization with linear partial criteria.

In this paper, the result mentioned above and some analogous results ([3] and [4]) are extended on a wide class of partial criteria.

Let $m>1$, $C=\{c_1, c_2, \ldots, c_m\}$ be a finite set. Suppose that each element c_j of the set C is weighted by the numbers $w_i(c_j)=a_{ij}, i \in N_n=\{1,2,\ldots,n\}$, where $A=\{a_{ij}\}_{n\times m} \in \boldsymbol{R}^{nm}$, $\boldsymbol{a}^i$ denoting the *i*-th string of the matrix *A*.

Let *T* be a system of nonempty subsets of the set *C*, $|T|>1$. All the elements of the set *T* are called trajectories.

Suppose that for any index $i \in N_n$ a real function $f_i(t,x)= = f(t, x_1, x_2, \ldots, x_m)$ is defined on the set *T* of trajectories and on the set $\boldsymbol{R}^m$ of vectors. Then the vector function (vector criterion)

$$f(t,A)=(f_1(t,\boldsymbol{a}^1), f_2(t,\boldsymbol{a}^2), \ldots, f_n(t,\boldsymbol{a}^n)) : T\times \boldsymbol{R}^{nm} \to \boldsymbol{R}^n$$

is given on the set *T*. Without loss of generality, we shall take the components of the vector criterion (partial criteria) for minimization under the fixed matrix *A*:

$$f_i(t,\boldsymbol{a}^i) \to \min_T, \quad i \in N_n.$$

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In these designations, the well-known partial criteria of the kinds MINSUM (linear) and MINMAX (bottle-neck) have the following form, respectively:

$$f_i(t,a^i) = \sum_{e_j \in t} a_{ij} \to \min_T,$$

$$f_i(t,a^i) = \max_{e_j \in t} a_{ij} \to \min_T.$$

We speak of vector (n-criteria) trajectorial problem to mean the problem of finding the Pareto set (set of efficient trajectories)

$$P(f,A) = \{t \in T : \forall\, t' \in T\,(\tau(t,t',A) \geq \mathbf{0} \Rightarrow \tau(t,t',A) = \mathbf{0})\},$$

where

$$\tau(t,t',A) = (\tau_1, \tau_2, \ldots, \tau_n),$$

$$\tau_i = \tau_i(t,t',a^i) = f_i(t,a^i) - f_i(t',a^i),\ i \in N_n,$$

$$\mathbf{0} = (0,0,\ldots,0) \in \boldsymbol{R}^n.$$

It is known that many discrete optimization problems on graphs, some scheduling problems and also the Boolean programming problems can be described as special cases of the single-criterion trajectorial problem (see [5]). For example, taking the edge set of a connected weighted graph as the set C and the Hamiltonian cycles set (the spanning treess set) as the set of trajectories T, with the linear objective we obtain the travelling salesman problem (the minimum spanning tree problem).

Following [3–6], we shall perturb the matrix $A \in \boldsymbol{R}^{nm}$, adding it to the matrices of the set

$$\mathrm{B}(\varepsilon) = \{B \in \boldsymbol{R}^{nm} : \|B\| < \varepsilon\},$$

where $\varepsilon > 0$, $\|\cdot\|$ is a norm defined in the space $\boldsymbol{R}^{nm}$ of $n \times m$-matrices.

The set $P(f,A)$ is called pseudo-stable if

$$\exists\, \varepsilon > 0\ \forall\ B \in \mathrm{B}(\varepsilon)\ (P(f,A) \supseteq P(f,A+B)).$$

Evidently, if there exists a trajectory $t \in T \setminus P(f,A)$ such that

(1) $$\forall\, \varepsilon > 0\ \exists\ B \in \mathrm{B}(\varepsilon)\ (t \in P(f,A+B)),$$

then the set $P(f,A)$ is not pseudo-stable.

The vector function $f(t,A)$ is said to possess α-property if the components $f_i(t,x), i \in N_n$, are continuous on $\boldsymbol{R}^m$ with respect to the variable x.

It can be easily shown that if the vector function $f(t,A)$ possesses α-property, then the following holds:

$$\exists\, \varepsilon > 0\ \forall\ B \in \mathrm{B}(\varepsilon)\ (Sl(f,A) \supseteq Sl(f,A+B)),$$

where $Sl(f,A) = \{t \in T : \forall\, t' \in T \setminus \{t\}\ \exists\, s \in N_n\ (\tau_s(t,t',a^s) \leq 0)\}$ is the Slater set, i.e., the set of weakly efficient trajectories (see [7] and [8]). This and the evident inclusion $P(f,A) \subseteq Sl(f,A)$ yield the following

THEOREM 1. *If the vector function* $f(t,A)$ *possesses* α*-property, then the equality* $P(f,A) = Sl(f,A)$ *is a sufficient condition of pseudo-stability of the Pareto set* $P(f,A)$.

By definition, put

$$N(t) = \{j \in N_m : c_j \in t\}.$$

The vector function $f(t,A)$ is said to possesses β-property if the following conditions hold for any index $i \in N_n$:

(β.1) for any trajectory $t \in T$, the function $f_i(t,x)$ is constant with respect to the variables $x_j, j \in N(E \setminus t)$, and grows with respect to the other variables $x_j, j \in N(t)$;

(β.2) for any trajectories t and t', the function $\tau_i(t,t',x)$ keeps their mark with respect to the variables $x_j, j \in N(t \cap t')$.

THEOREM 2. *If the vector function* $f(t,A)$ *possesses* β*-property, then the equality* $P(f,A) = Sl(f,A)$ *is a necessary condition of stability of the Pareto set* $P(f,A)$.

Proof. Assume that the vector function $f(t,A)$ possesses β-property and the Pareto set $P(f,A)$ is stable. Suppose the opposite: $P(f,A) \neq Sl(f,A)$, i.e., there exists a trajectory $t \in Sl(f,A) \setminus P(f,A)$.

By definition of the set $Sl(f,A)$, for any trajectory $t' \in T \setminus \{t\}$ there exists an index $s \in N_n$ such that $\tau_s(t,t'a^s) \leq 0$. For any positive number ε, there exists a number $\delta > 0$ such that the perturbing matrix $B = \{b_{ij}\}_{n \times m}$ with the elements

$$b_{ij} = \begin{cases} -\delta, & \text{if}\quad i \in N_n,\quad j \in N(t), \\ \delta & \text{if}\quad i \in N_n,\quad j \in N(E \setminus t) \end{cases}$$

belongs to the set $\mathrm{B}(\varepsilon)$. In view of condition (β.1), we have

$$\tau_s(t,t'a^s + b^s) = \tau_s(t,t',a^s + c + d),$$

where

$$c = (c_1, c_2, \ldots, c_m) \in \boldsymbol{R}^m, c_j = \begin{cases} -\delta, & \text{if} \quad j \in N(t \cap t'), \\ 0 & \text{otherwise}, \end{cases}$$

$$\boldsymbol{d} = (d_1, d_2, \ldots, d_m) \in \boldsymbol{R}^m, d_j = \begin{cases} -\delta, & \text{if} \quad j \in N(t \setminus t'), \\ \delta & \text{if} \quad j \in N(t' \setminus t), \\ 0 & \text{otherwise}. \end{cases}$$

Using (β.2), we get $\tau_s(t, t' \boldsymbol{a}^s + c) \le 0$. Since one of the sets $t \setminus t', t' \setminus t$ is nonempty, by condition (β.1) we obtain $\tau_s(t, t' \boldsymbol{a}^s + c + d) < 0$.

Thus, we have

$$\forall \varepsilon > 0 \; \exists \; B \in \mathrm{B}(\varepsilon) \; \forall \, t' \in T \setminus \{t\} \; \exists \, s \in N_n (\tau_s(t, t' \boldsymbol{a}^s + \boldsymbol{b}^s) < 0),$$

i.e.,

$$\forall \varepsilon > 0 \; \exists \; B \in B(\varepsilon) \, (t \in P(f, A + B)).$$

Hence, the set $P(f, A)$ is not pseudo-stable (see (1)). The contradiction proves Theorem 2.

Theorems 1 and 2 yield the following result.

COROLLARY 1. *Assume that the vector function $f(t, A)$ possesses α-property and β-property. Then the Pareto set $P(f, A)$ is pseudo-stable iff the equality $P(f, A) = Sl(f, A)$ holds.*

The set $P(f, A)$ is called quasi-stable if

$$\exists \varepsilon > 0 \; \forall \; B \in \mathrm{B}(\varepsilon) \, (P(f, A) \subseteq P(f, A + B)).$$

Evidently, if the vector function $f(t, A)$ possesses α-property, then the following statement holds:

$$\exists \varepsilon > 0 \; \forall \; B \in \mathrm{B}(\varepsilon) \, (Sm(f, A) \subseteq Sm(f, A + B)),$$

where $Sm(f, A) = \{t \in T : \forall \, t' \in T \setminus \{t\} \; \exists \, i \in N_n \; (\tau_i(t, t', \boldsymbol{a}^i) < 0)\}$ is the Smale set, i.e., the set of strongly efficient trajectories (see [9]). This and the evident inclusion $Sm(f, A) \subseteq P(f, A)$ yield the following result.

THEOREM 3. *If the vector function $f(t, A)$ possesses α-property, then the equality $P(f, A) = Sm(f, A)$ is a sufficient condition of quasi-stability of the Pareto set $P(f, A)$.*

Let $i \in N_n$. If for any trajectory $t \in T$ the function $f_i(t, x)$ has no constant intervals with respect to the variables $x_j, j \in N(t)$, and this function is constant with respect to the other variables $x_j, j \in N(E \setminus t)$, then the function $f_i(t, x)$ is called special.

The following lemma is evident.

LEMMA. *Let $t, t' \in T, p \in N(t \setminus t'), s \in N_n$. If the function $f_s(t, x)$ is special, then*

$$\text{(2)} \qquad \forall \, \boldsymbol{x} \in \boldsymbol{R}^m \;\; \forall \varepsilon > 0 \; \exists \, \delta \in (-\varepsilon; \varepsilon) \, (\tau_s(t, t', \boldsymbol{x} + \delta \boldsymbol{e}_p) \ne 0),$$

where $\boldsymbol{e}_p$ is the p-th basis vector of $\boldsymbol{R}^m$.

The vector function $f(t, A)$ is said to possess γ-property if there exists a special function among the components of $f(t, A)$.

REMARK 1. *It is evident that the vector function $f(t, A)$ possesses α-property, β-property and γ-property if*

$$\forall \, i \in N_n \;\; f_i(t, x) = g\Big(\sum_{j \in N(t)} f_{ij}(x_j)\Big),$$

where $g(x)$ and $f_{ij}(x)$ are continuous growing functions.

If all the partial criteria are of MINMAX kind, then the vector function $f(t, A)$ possesses α-property only.

THEOREM 4. *If the vector function $f(t, A)$ possesses γ-property, then the inequality $P(f, A) = Sm(f, A)$ is a necessary condition of quasi-stability of the Pareto set $P(f, A)$.*

Proof. Assume that the function $f_s(t, x)$ is special and the Pareto set $P(f, A)$ is quasi-stable. Suppose the opposite: $P(f, A) \ne Sm(f, A)$, i.e., there exists an efficient trajectory t that is not strongly efficient. Then there exists a trajectory $t' \in P(f, A) \setminus \{t\}$ such that $\tau(t, t', A) = 0$. Without loss of generality, put $t \setminus t' \ne \varnothing$, $p \in N(t \setminus t')$. Using the lemma, we get (2), i.e., $\tau_s(t, t' \boldsymbol{a}^s + \boldsymbol{b}^s) \ne 0$, where $\boldsymbol{b}^s$ is the s-th string of the matrix $B = \{b_{ij}\}_{n \times m}$ with the elements

$$b_{ij} = \begin{cases} 0, & \text{if} \quad (i, j) \ne (s, p), \\ \delta, & \text{if} \quad (i, j) = (s, p). \end{cases}$$

Taking into account the equality $\tau(t, t', A) = 0$, we obtain

$$\forall i \neq s \ (\tau_i(t, t', a^i + b^i) = 0),$$

$$\tau_s(t, t', a^s + b^s) \neq 0.$$

Hence, either $t \notin P(f, A+B)$ or $t' \notin P(f, A+B)$. If we recall the memberships $t, t' \in P(f, A)$, we see that the Pareto set $P(f, A)$ is not quasi-stable. The contradiction proves Theorem 2.

Theorems 3 and 4 imply the following

COROLLARY 2. *Assume that the vector function $f(t, A)$ possesses α-property and γ-property. Then the Pareto set $P(f, A)$ is quasi-stable iff the equality $P(f, A) = Sm(f, A)$ holds.*

The Pareto set $P(f, A)$ is called stable if

$$\exists \varepsilon > 0 \ \forall B \in \mathrm{B}(\varepsilon) \ (P(f, A) = P(f, A+B)).$$

Clearly, the set $P(f, A)$ is stable iff this set is pseudo-stable and quasi-stable.

Theorems 1 and 3 yield the following

THEOREM 5. *If the vector function $f(t, A)$ possesses α-property, then the relation $Sm(f, A) = P(f, A) = Sl(f, A)$ is a sufficient condition of stability of the Pareto set $P(f, A)$.*

Evidently if the vector function $f(t, A)$ possesses β-property, then the vector function possesses γ-property. Using this, Theorem 2 and Theorem 4, we obtain the following result.

THEOREM 6. *If the vector function $f(t, A)$ possesses β-property, then the relation $Sm(f, A) = P(f, A) = Sl(f, A)$ is a necessary condition of stability of the Pareto set $P(f, A)$.*

From Theorem 5 and Theorem 6 we obtain a necessary and sufficient condition of stability of the Pareto set that is formulated in the beginning of the paper.

COROLARY 3. *Assume that the vector function $f(t, A)$ possesses α-property and β-property. Then the Pareto set $P(f, A)$ is stable iff the equalities $Sm(f, A) = P(f, A) = Sl(f, A)$ hold.*

REMARK 2. *Evidently, the afore-mentioned results are valid in the case when the functions $f_i(t, x)$, $i \in N_n$, are defined on arbitrary open subsets of $\boldsymbol{R}^m$.*

REMARK 3. *It is easy to show that the restrictions imposed on the vector function $f(t, A)$ in Theorems 1–6 cannot be omitted unconditionally.*

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