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ON SOME PROPERTIES OF STANCU OPERATOR

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1. INTRODUCTION

The positive linear polynomial operator defined by

$$P_n^{\alpha}(f,x)=\sum_{k=0}^{n} f(k/n)\,w_{n,k}(x,\alpha),\quad n\in N,\ x\in[0,1],\ \alpha\geq 0, \tag{1}$$

where f is a real function on [0, 1] and

$$w_{n,k}(x,\alpha)=\binom{n}{k}\frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=0}^{n-k-1}(1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)},$$

was introduced by Stancu [6], who studied, among other properties, the convergence of $P_n^{\alpha} f$ to f as $n\to\infty$ and $0\leq\alpha=\alpha(n)\to 0$.

In the case $\alpha=0$, P_n^0 is the Bernstein operator B_n given by

$$B_n(f,x)=\sum_{k=0}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}, \tag{2}$$

where $f\in C[0,1]$. Lorentz [4, p. 102] proved that for $f\in C[0,1]$:

$$|B_n(f,x)-f(x)|\leq M\cdot\frac{x(1-x)}{n}\quad\text{if and only if}\quad \omega^2(f,h)=O(h^2). \tag{3}$$

Here $\omega^2(f,t)$ is the classical modulus of smoothness defined by

$$\omega^2(f,t)=\sup_{0<h\leq t}\|\Delta_h^2 f(x)\|_{C[0,1]}, \tag{4}$$

where

$$\Delta_h^2 f(x)=\begin{cases} f(x+h)-2f(x)+f(x-h), & \text{if}\quad x\pm h\in[0,1];\\ 0, & \text{otherwise.}\end{cases}$$

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In this paper, we provide further approximation properties for P_n^α. Let φ denote the following function: $\varphi(x)=\sqrt{x(1-x)}, x\in[0,1]$. For a function $g\in C[0,1]$ we denote the uniform norm on a subinterval $[a,b]\subseteq[0,1]$ by $\|g\|_{C[a,b]}=\sup\{|g(x)|: x\in[a,b]\}$.

2. MAIN RESULTS

The theorems in question can be stated as follows:

THEOREM 1. *Let* $\alpha(n)=o(n^{-1}), \alpha(n)\cdot n\le 1$, *for* $n=1,2,\dots$ *and* $f\in C[0,1]$. *Then for each* $M>0$,

(5) $$|P_n^\alpha(f,x)-f(x)|\le M\frac{\varphi^2(x)}{n},\quad x\in[0,1],\quad n=1,2,\dots$$

holds exactly when $\omega^2(f,h)\le Mh^2, h>0$.

THEOREM 2. *Let* $0\le\alpha(n)\to 0\ (n\to\infty)$ *and* $f\in C[0,1]$ *with* $\omega^2(f,h)\le Mh^\beta$, $h>0$, *and* $0<\beta<2$. *Then*

$$|P_n^\alpha(f,x)-f(x)|\le M\left(\frac{\varphi^2(x)}{n}\right)^{\beta/2},\quad x\in[0,1],\quad n=1,2,\dots$$

THEOREM 3. *Let* $\alpha=\alpha(n)=O(n^{-1}), \alpha_n^2=(1+n\alpha)/(n(1+\alpha)),\quad n=1,2,\dots$ *and* $f\in C[0,1]$. *Then there exists* $M>0$ *such that*

(6) $$\|P_n^\alpha f-f\|_{C[0,1]}\le M\cdot\omega_\varphi^2(f,\alpha_n)_{C[0,1]},$$

where $\omega_\varphi^2(f,t)_{C[0,1]}=\sup\limits_{0<h\le t}\|\Delta_{h\varphi(x)}^2 f(x)\|_{C[0,1]}$ *is the Ditzian-Totik modulus of smoothness* [3].

THEOREM 4. *Let* $\alpha=\alpha(n)=O(n^{-1})$, *and* $0<\beta<2$. *If* $f\in C[0,1]$ *and* $\omega_\varphi^2(f,h)_{C[0,1]}=O(h^\beta)$, *then* $\|P_n^\alpha f-f\|_{C[0,1]}=O(n^{-\beta/2})$.

In order to prove theorems, we need some lemmas first.

3. LEMMAS

For proving Theorem 1 we need the following lemmas:

LEMMA 1 *(the localization theorem). If* $f\in C[0,1]$ *vanishes on a subinterval* $[a,b]\subseteq[0,1]$ *and* $\alpha(n)=o(n^{-1})$, *then*

(7) $$P_n^\alpha(f,x)=o(n^{-1}),\quad x\in(a,b).$$

Proof. Suppose that f has second derivative $f''(x)$ for some $x\in[0,1]$. Then, by Theorem 7.1 [6, p. 1192] and $\alpha(n)=o(n^{-1})$, we have

(8) $$\lim_{n\to\infty} n[P_n^\alpha(f,x)-f(x)]=\frac{x(1-x)}{2}f''(x).$$

From Lemma 5.2 [1, p. 134] we get (7).

LEMMA 2. *If* $f\in C[0,1]$, $\alpha(n)=o(n^{-1})$ *and*

(9) $$\limsup_{n\to\infty}\{n[P_n^\alpha(f,x)-f(x)]\}\ge 0,\quad x\in(a,b)\subseteq[0,1],$$

then f is convex on [0, 1].

Proof. Suppose that f is not convex. Then, using the parabola technique [1, p. 124], there exist $x_0\in(a,b), \delta>0$ and $Q(x)=cx^2+dx+c, c<0$ such that $f(x_0)=Q(x_0)$ and $f(x_0+t)\le Q(x_0+t)$ for all t, $|t|\le\delta$. We can extend f to a function $F\in C[0,1]$ so that $F(x)=f(x)$, $x\in[x_0-\delta,x_0+\delta]$, $F(x)\le Q(x)$, $x\in[0,1]$. By Lemma 4.1. [6, p. 1184]

$$P_n^\alpha(1,x)=1,\quad P_n^\alpha(u,x)=x$$

and

$$P_n^\alpha(u^2,x)=\frac{1}{1+\alpha}\left[\frac{x(1-x)}{n}+x(x+\alpha)\right],$$

so we obtain

$$n[P_n^\alpha(F,x_0)-F(x_0)]\le n[P_n^\alpha(Q,x_0)-Q(x_0)]$$

$$=c\cdot x_0(1-x_0)\cdot\frac{1+n\alpha}{1+\alpha}<0.$$

But Lemma 1 shows that

$$n[P_n^\alpha(f,x_0)-f(x_0)]\le c\cdot x_0(1-x_0)\cdot\frac{1+n\alpha}{1+\alpha}+o(1),$$

which contradicts the assumption (9).

In order to prove Theorem 2 we need some other lemmas:

LEMMA 3. *For* $f\in C[0,1]$ *we have* $\|P_n^\alpha f\|_{C[0,1]}\le\|f\|_{C[0,1]}$.

Proof. Since $\sum\limits_{k=0}^n w_{n,k}(x,\alpha)=1$ [6, p. 1184], we have

$$|P_n^\alpha(f,x)|\le\sum_{k=0}^n w_{n,k}(x,\alpha)|f(k/n)|\le\|f\|_{C[0,1]}.$$

So

$$\|P_n^\alpha f\|_{C[0,1]} \le \|f\|_{C[0,1]}.$$

LEMMA 4. *For some constant $C>0$ and all $f \in C[0,1]$,*

(10) $$|P_n^\alpha(f,x)-f(x)| \le C\omega^2(f,\varphi(x)/\sqrt{n}), \quad x\in[0,1].$$

Proof. Let $f \in C^2[0,1]$. By Taylor's formula:

(11) $$f\left(\frac{k}{n}\right)=f(x)+\left(\frac{k}{n}-x\right)f'(x)+\left(\frac{k}{n}-x\right)^2\left[\frac{1}{2}f''(x)+g\left(\frac{k}{n}-x\right)\right],$$

where $g(y)=g_x(y)\to 0$ as $y\to 0$. This shows that the last term in (11) does not exceed $\|f''\|_{C[0,1]}(k/n-x)^2/2$. Then

(12) $$\|P_n^\alpha f-f\|_{C[0,1]} \le \|f''\|_{C[0,1]}\cdot\frac{x(1-x)}{2n}\cdot\frac{1+n\alpha}{1+\alpha}, \quad f\in C^2[0,1].$$

Using Lemma 3, we obtain the boundedness of the operator P_n^α and, by (12) and Theorem 5.3 [2, p. 218], we obtain (10).

Remark. For $\alpha=0$ the inequality (10) reduces to the known inequality of Popoviciu [5] corresponding to the Bernstein polynomial (2):

(13) $$|B_n(f,x)-f(x)| \le C\omega^2(f,\varphi(x)/\sqrt{n}), \quad x\in[0,1].$$

Theorem 4 requires the following lemmas:

LEMMA 5. *For* $R_2(f,u,x)=\int_x^u (u-v)f''(v)\,dv$ *we have*

(14) $$|R_2(f,u,x)| \le \frac{|u-x|}{\varphi^2(x)}\cdot\left|\int_u^x \varphi^2(v)|f''(v)|\,dv\right|$$

for $x,u\in[0,1]$.

Proof. Lemma 5 is a particular case of Lemma 9.6.1 [3, p. 140].

LEMMA 6. *For $A>0$ a fixed number and f a differentiable function on* [0, 1] *with f' locally absolutely continuous in* [0, 1] *we have*

(15) $$\|P_n^\alpha f-f\|_{C[A/n,1-A/n]} \le \frac{1+n\alpha}{n(1+\alpha)}\cdot\|\varphi^2 f''\|_{C[0,1]}.$$

Proof. We consider the maximal function of Hardy-Littlewood

$$M(G,x)=\sup_u\left|\frac{1}{x-u}\cdot\int_u^x G(v)\,dv\right|,$$

where $G(v)=\varphi^2(v)f''(v)$. By Lemma 5 we obtain

(16) $$\|P_n^\alpha(R_2(f,\cdot,x),x)\|_{C[A/n,1-A/n]} \le$$
$$\le \left\|P_n^\alpha\left(\frac{(\cdot-x)^2}{\varphi^2(x)},x\right)\right\|_{C[A/n,1-A/n]}\cdot\|M(G,x)\|_{C[0,1]}$$

and

(17) $$\|M(G,x)\|_{C[0,1]} \le \|\varphi^2 f''\|_{C[0,1]}.$$

Since

$$P_n^\alpha((u-x)^2\cdot\varphi^{-2}(x),x)=\frac{1+n\alpha}{n(1+\alpha)},$$

we have

(18) $$\|P_n^\alpha(R_2(f,\cdot,x),x\|_{C[A/n,1-A/n]} \le \frac{1+n\alpha}{n(1+\alpha)}\cdot\|\varphi^2 f''\|_{C[0,1]},$$

using (16) and (17). By Taylor's formula:

$$f(u)=f(x)+f'(x)(u-x)+\int_x^u (u-v)f''(v)\,dv.$$

Then

$$P_n^\alpha(f(\cdot),x)=f(x)+f'(x)P_n^\alpha((\cdot-x),x)+P_n^\alpha(R_2(f,\cdot,x),x).$$

Hence, by (18), we obtain (15).

4. PROOFS

Proof of Theorem 1. If

$$|P_n^\alpha(f,x)-f(x)| \le M\cdot\frac{\varphi^2(x)}{n}, \quad x\in[0,1],\ n=1,2,3,\ldots$$

is satisfied, we put $g(u)=f(u)+Mu^2/2$ and obtain

$$n[P_n^\alpha(g,x)-g(x)] \ge M\frac{x(1-x)}{2}\cdot\frac{(n-1)\alpha}{1+\alpha} \ge 0.$$

Then, by Lemma 2, g is convex on [0, 1] and, therefore, $g(x+h)-2g(x)+g(x-h)\ge 0$. Hence $f(x+h)-2f(x)+f(x-h)\ge -Mh^2$. Analogously, for $g(u)=-f(u)+Mu^2/2$

we have $f(x+h)-2f(x)+f(x-h)\le Mh^2$, so $|\Delta_h^2 f(x)|\le Mh^2$. Therefore $\omega^2(f,h)\le Mh^2$, $h>0$.

Conversely, let $f\in C[0,1]$ with $\omega^2(f,h)\le Mh^2$, $h>0$. Then f belongs to the Lipschitz space $Lip(2, C[0,1])$, which implies, in view of Theorem 9.3 [2, p. 53], that there exists f' on [0, 1] and f' is absolutely continuous with $|f''(x)|\le M$ a.e. $x\in[0,1]$.

Let $x\in[0,1]$ be a fixed point. For the linear function $l(y)=f(x)+f'(x)\cdot(x-y)$, by Taylor's formula, we have $|f(y)-l(y)|\le M(y-x)^2/2$. Since P_n^α preserves linear functions, we obtain

$$|P_n^\alpha(f,x)-f(x)|=|P_n^\alpha(f-l,x)|\le$$

$$\le\frac{M}{2}\cdot P_n^\alpha((\cdot-x)^2;x)=M\cdot\frac{x(1-x)}{2n}\cdot\frac{1+n\alpha}{1+\alpha}.$$

By the hypothesis $n\cdot\alpha(n)\le 1$, for $n=1,2,\dots$ we obtain

$$|P_n^\alpha(f,x)-f(x)|\le M\cdot\frac{\varphi^2(x)}{n},\quad x\in[0,1],\quad n=1,2,\dots,$$

Proof of Theorem 2. Lemma 4 and $\omega^2(f,h)\le Mh^\beta$, $0<\beta<2$, imply the inequality $|P_n^\alpha(f,x)-f(x)|\le M(\varphi^2(x)/n)^{\beta/2}$.

Proof of Theorem 3. We have

$$P_n^\alpha(1,x)=1,\qquad P_n^\alpha(u,x)=x$$

and

$$P_n^\alpha(u^2,x)=\frac{1}{1+\alpha}\left[\frac{x(1-x)}{n}+x(x+\alpha)\right].$$

Then

$$P_n^\alpha((u-x)^2,x)=x(1-x)\cdot\frac{1+n\alpha}{n(1+\alpha)}=\varphi^2(x)\cdot\alpha_n^2,$$

for $n=1,2,\dots$ So, our statement is a direct consequence of Theorem 1 [7, p. 165].

Proof of Theorem 4. We can choose $g_n\in C[0,1]$ such that $\varphi^2 g_n''\in C[0,1]$ and g_n' is locally absolutely continuous in [0, 1], which satisfies

(19) $$\|f-g_n\|_{C[0,1]}\le 2K_{2,\varphi}(f,n^{-1})_{C[0,1]}$$

and

(20) $$\|\varphi^2 g_n''\|_{C[0,1]}\le 2n\,K_{2,\varphi}(f,n^{-1})_{C[0,1]}.$$

By $K_{2,\varphi}(f,t^2)_{C[0,1]}$ we denote the K-functional of the pair of spaces $C[0,1]$ and a corresponding weighted Sobolev space with weight function φ^2 given by

$$K_{2,\varphi}(f,t^2)_{C[0,1]}=\inf_g\{\|f-g\|_{C[0,1]}+t^2\|\varphi^2 g''\|_{C[0,1]}:g'\in A.C._{loc}\},$$

where $g'\in A.C._{loc}$ means that g is differentiable and g' is absolutely continuous in every closed interval $[a,b]\subseteq[0,1]$. Using Lemma 3 and Lemma 6 and writing $f=f-g_n+g_n$, we estimate

(21) $$\|P_n^\alpha(f-g_n)-(f-g_n)\|_{C[0,1]}\le C\|f-g_n\|_{C[0,1]}$$

and

(22) $$\|P_n^\alpha g_n-g_n\|_{C[A/n,1-A/n]}\le\frac{1+n\alpha}{n(1+\alpha)}\cdot\|\varphi^2 g''\|_{C[0,1]}.$$

By Theorem 7.2.1 and Theorem 7.3.1 [3, p. 79 and p. 84, respectively], we can choose g_n to be $P_{[\sqrt{n}]}$ the best $[\sqrt{n}]$-*th* degree polynomial approximation in $C[0,1]$. We have

(23) $$\|P_n^\alpha P_{[\sqrt{n}]}-P_{[\sqrt{n}]}\|_{C[0,1]}\le\overline{M}\|P_n^\alpha P_{[\sqrt{n}]}-P_{[\sqrt{n}]}\|_{C[A/n,1-A/n]},$$

using Theorem 8.4.8 [3, p. 108] translated from [–1, 1] to [0, 1] and n to $[\sqrt{n}]$. But, in view of Theorem 2.1.1 [3, p. 11], there exists $M>0$ such that

(24) $$M^{-1}\omega_\varphi^2(f,n^{-1/2})_{C[0,1]}\le K_{2,\varphi}(f,n^{-1})_{C[0,1]}\le M\omega_\varphi^2(f,n^{-1/2})_{C[0,1]}.$$

Then, by (23), (24), (22) and (20), we obtain

(25) $$\|P_n^\alpha P_{[\sqrt{n}]}-P_{[\sqrt{n}]}\|_{C[0,1]}\le\overline{M}\cdot\frac{1+n\alpha}{n(1+\alpha)}\cdot 2n\cdot K_{2,\varphi}(f,n^{-1})_{C[0,1]}\le$$

$$\le M'\cdot\frac{1+n\alpha}{1+\alpha}\cdot\omega_\varphi^2(f,n^{-1/2})_{C[0,1]}.$$

On the other hand, by (21), (19) and (24), we have

(26) $$\|P_n^\alpha(f-P_{[\sqrt{n}]})-(f-P_{[\sqrt{n}]})\|_{C[0,1]}\le$$

$$\le 2CK_{2,\varphi}(f,n^{-1})_{C[0,1]}\le M''\cdot\omega_\varphi^2(f,n^{-1/2})_{C[0,1]}.$$

Consequently, from (25) and (26), we obtain

$$\| P_n^\alpha f - f \|_{C[0,1]} \le \left(M' \cdot \frac{1+n\alpha}{1+\alpha} + M'' \right) \cdot \omega_\varphi^2 (f, n^{-1/2})_{C[0,1]}.$$

Using the hypotheses $\alpha = O(n^{-1})$ and $\omega_\varphi^2 (f, n^{-1/2})_{C[0,1]} = O(h^\beta)$, $0 < \beta < 2$, we obtain $\| P_n^\alpha f - f \|_{C[0,1]} = O(n^{-\beta/2})$. Thus the theorem is proved completely.

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