REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome XXVII, N° 1, 1998, pp. 107–115

MEAN-VALUE FORMULAE FOR INTEGRALS INVOLVING GENERALIZED ORTHOGONAL POLYNOMIALS

LAURA GORI, D. D. STANCU

1. It is known that if we have a sequence of orthogonal polynomials (p_m), associated with a nonnegative measure $d\alpha$ on an interval (a, b) of the real line and if we consider a function $f \in C^m(a,b)$, then we can write the following mean-value formula, of N. Cioranescu [3], for integrals:

$$\int_a^b f(x)\, p_m(x)\, d\alpha(x) = \frac{f^{(m)}(\xi)}{m!} \int_a^b x^m p_m(x)\, d\alpha(x), \quad a < \xi < b. \tag{1}$$

In this paper we give several extensions of it to some classes of nonclassical orthogonal polynomials, including the power-orthogonal polynomials corresponding to a measure of the form $\omega d\alpha$, where ω is a nonnegative polynomial having given real multiple zeros.

It is remarkable the special case of s-orthogonal polynomials $P_{m,s}$, for which $\int_a^b P_{m,s}^{2s+2}(x)\, d\alpha(x) = \min$, when we obtain the extension

$$\int_a^b f(x)\, P_{m,s}^{2s+1}(x)\, d\alpha(x) = \frac{f^{(m)}(\xi)}{m!} \int_a^b x^m P_{m,s}^{2s+1}(x)\, d\alpha(x). \tag{2}$$

It reduces to formula (1) when $s = 0$ (the case of ordinary orthogonal polynomials).

2. We start from a "method of parameters" (see [20]) for constructing a general Gauss-Christoffel quadrature formula by using multiple preassigned nodes and multiple free nodes.

We assume that $\alpha(x)$ has infinitely many points of increase and that $d\alpha(x)$ has finite moments of all orders.

AMS Subject Classification: 26A24, 33C25, 41A55, 65D32.

Let a_i be fixed nodes from the interval (a, b), given with their orders of multiplicities r_i, such that we have on (a, b)

(3) $$\omega(x) = \varepsilon \prod_{i=1}^{n} (x-a_i)^{r_i} \geq 0 \quad \text{for} \quad \varepsilon = 1 \quad \text{or} \quad \varepsilon = -1.$$

One further considers m multiple free nodes $x_1, \ldots, x_m$, such that $a < x_1 < \ldots < x_m < b$, their orders of multiplicities being, respectively, the given odd positive integers $2s_1+1, \ldots, 2s_m+1$.

Let u be the polynomial of the free nodes

(4) $$u(x) = \prod_{k=1}^{m} (x-x_k)^{2s_k+1}.$$

In 1959 the second author [20] constructed and investigated a general quadrature formula, for weighted integrals

$$I(g) = I(g; d\alpha) = \int_a^b g(x)\, d\alpha(x),$$

by using preassigned multiple nodes and multiple free nodes, of the following form

(5) $$I(g; d\alpha) = \phi(g) + R(g; d\alpha),$$

where

(6) $$\phi(f) = \sum_{i=1}^{n} \sum_{j=0}^{r_i-1} A_{i,j} g^{(j)}(a_i) + \sum_{k=1}^{m} \sum_{h=0}^{2s_k} B_{k,h} g^{(h)}(x_k),$$

the nodes x_k being selected such that formula (5)–(6) has the highest degree of exactness, that is, to have $R(e_r; d\alpha) = 0$ for $r = 0, 1, \ldots, D$, where e_r is the monomial $e_r(x) = x^r$ and D is as large as possible.

If at (5) we have an interpolatory quadrature formula, then we have $M \leq D$, where

(7) $$M = \sum_{i=1}^{n} r_i + \sum_{k=1}^{m} (2s_k+1) - 1$$

is the degree of the Lagrange-Hermite interpolation polynomial associated to the function g and the multiple nodes a_i and x_k.

If we replace in (5) $g = \omega U$, where

$$U(x) = u(x)(x-x_1)\ldots(x-x_m) = \prod_{k=1}^{m} (x-x_k)^{2s_k+2},$$

we obtain

$$R(\omega U; d\alpha) = \int_a^b \omega(x)\, U(x)\, d\alpha(x) > 0.$$

Since the degree of ωU is $M+m+1$, we can conclude that the highest degree of exactness of (5) must satisfy the inequality: $D \leq M+m$.

In order to construct such a quadrature formula, we have used a method of additional arbitrary nodes γ_j $(j = 1, 2, \ldots, m)$, such that a_i, x_k and γ_j are distinct points from (a, b). Let $v(x) = (x-\gamma_1)\ldots(x-\gamma_m)$.

The corresponding Lagrange-Hermite interpolation formula is of the form

$$g(x) = (L_H g)(x) + (rg)(x),$$

where we have

(8) $$(L_H g)(x) = (L_H g)\left(x; \begin{matrix} a_i \\ ; \\ r_i \end{matrix} \begin{matrix} x_k \\ \\ 2s_k+1 \end{matrix}; \gamma_j\right)$$

and

(9) $$(rg)(x) = \omega(x)\, u(x)\, v(x) \left[\begin{matrix} a_i \\ ; \\ r_i \end{matrix} \begin{matrix} x_k \\ \\ 2s_k+1 \end{matrix}; \gamma_j; x; g\right],$$

the brackets used in the remainder representing the symbol for divided differences.

It is easily verified that we have (see [21])

$$(L_H g)(x) = v(x)(L_H g_1)\left(x; \begin{matrix} a_i \\ ; \\ r_i \end{matrix} \begin{matrix} x_k \\ \\ 2s_k+1 \end{matrix}\right) + \omega(x)\, u(x)\, (L g_2)(x; \gamma_j),$$

where $g_1 = f/v$, $g_2 = f/(\omega u)$.

By integrating the preceding interpolation formula, we obtain a quadrature formula of the form

(10) $$I(g; d\alpha) = \phi(g) + \Omega(g) + R(g; d\alpha),$$

where

(11) $$\Omega(g) = \sum_{j=1}^{m} D_j g(\gamma_j), \quad R(g; d\alpha) = I(rg; d\alpha).$$

Because the divided difference which occurs in the remainder is of order $M+m+1$, it follows that the quadrature formula (10) has the degree of exactness $D = M+m$.

Now we seek to determine the nodes x_k, having respectively the given orders of multiplicities $2s_k+1$, such that $D_1 = \ldots = D_m = 0$. Because

$$D_j = \int_a^b \frac{\omega(x)\, u(x)\, v_j(x)}{\omega(\gamma_j)\, u(\gamma_j)\, v_j(\gamma_j)}\, d\alpha(x), \quad v_j(x) = \frac{v(x)}{x-\gamma_j},$$

it follows that we should have

$$I(\omega u v_j; d\alpha) = 0 \quad (j = 1, 2, \ldots, m).$$

Taking into account that γ_j are arbitrary, it follows that $D_j = 0$ if and only if the polynomial u is orthogonal on (a, b), with respect to the distribution $\omega d\alpha$, to all polynomials of degree at most $m-1$.

Consequently, we arrive at the following system of m equations in m unknowns $x_1, \ldots, x_m$:

(12) $$\int_a^b \omega(x)\, u(x)\, x^k\, d\alpha(x) = 0 \quad (k = 0, 1, \ldots, m-1).$$

The solution of the system (12) identifies with the solution of the following extremal problem:

(13) $$I(U; d\alpha) = \int_a^b \omega(x)\, (x - x_1)^{2s_1+2} \ldots (x - x_m)^{2s_m+2}\, d\alpha(x) = \min.$$

The nodes x_k, with given odd orders of multiplicities, which are determined by (12) or by (13), will be called Gaussian nodes, corresponding to the measure $\omega(x)\, d\sigma(x)$.

3. Given the sequence of nonnegative integers $\sigma = (s_1, \ldots, s_m)$ and the measure $\omega d\alpha$, the relations (12) permit us to define a sequence of polynomials

(14) $$P_{k,\sigma}(x) = (x - x_{1,\sigma}) \ldots (x - x_{k,\sigma}), \quad a < x_{1,\sigma} < \ldots < x_{k,\sigma} < b$$

such that

$$\int_a^b \omega(x)\, P_{k,\sigma}(x)\, Q_{m,\sigma}(x)\, d\alpha(x) = 0 \quad (0 \le k \le m-1),$$

where

(15) $$Q_{m,\sigma}(x) = \prod_{j=1}^{m} (x - x_{j,\sigma})^{2s_j+1}.$$

The polynomials $P_{k,\sigma}$ represent a sequence of σ-orthogonal polynomials, corresponding to the measure $\omega d\alpha$ and the interval (a, b).

Such a new type of orthogonality was considered in many papers: [24], [2], [17], [5], [7], [8] and [15]. In [10] it was given a stable procedure for the numerical construction of the σ-orthogonal polynomials.

4. Once the nodes $x_k = x_{k,\sigma}$ are determined, we can see that formula (10) reduces to the form (5)–(6). In [20] there was proved that the coefficients of formula (10) do not depend on the nodes γ_j. Since the remainder $R(g; d\alpha)$

should also be independent of these parameters, let us make $\gamma_j \to x_j$ $(j = 1, 2, \ldots, m)$ in the remainder given at (9), (10) and (11). We obtain

(16) $$R(g; d\alpha) = \int_a^b \omega(x)\, U(x)\, D_n(g; x)\, d\alpha(x),$$

where

(17) $$D_1(g; x) = \left[x; \begin{matrix} a_i \\ r_i \end{matrix}; \begin{matrix} x_k \\ 2x_k + 2 \end{matrix}; g(t)\right].$$

Because $\omega(x)\, U(x) \ge 0$ on (a, b), we can apply the mean-value theorem of the integral calculus and we get

$$R(g; d\alpha) = D_1(g; \eta) \int_a^b \omega(x)\, U(x)\, d\alpha(x), \quad \eta \in (a, b).$$

Assuming that $f \in C^{N+1}(a, b)$, where $N = M + m$, we can use the mean-value theorem of divided differences and we find

$$R(g; d\alpha) = \frac{g^{(N+1)}(\xi)}{(N+1)!} \int_a^b \omega(x)\, U(x)\, d\alpha(x).$$

5. The quadrature rule (10), containing the parameters $\gamma_1, \ldots, \gamma_m$, can be used for obtaining generalizations of the mean-value formula (1) of N. Cioranescu.

Indeed, if we replace $g = \omega f u$, we obtain

$$I(\omega f u; d\alpha) = \int_a^b \omega(x)\, u(x)\, v(x)\, \Omega(\omega f u; x)\, d\alpha(x),$$

where

$$\Omega(\omega f u; x) = \left[x; \begin{matrix} a_i \\ r_i \end{matrix}; \begin{matrix} x_k \\ 2s_k + 1 \end{matrix}; \gamma_1, \ldots, \gamma_m; \omega(t) f(t) u(t)\right].$$

But, according to an "absorption formula" from the theory of divided differences, we can write

$$\Omega(\omega f u; x) = [x, \gamma_1, \ldots, \gamma_n; f(t)].$$

Hence we have

$$I(\omega f u; d\alpha) = \int_a^b \omega(x)\, u(x)\, v(x)\, [x, \gamma_1, \ldots, \gamma_m; f(t)]\, d\alpha(x).$$

Because the second side of this equality must also be independent of the parameters $\gamma_1,\ldots,\gamma_m$, we shall consider the limiting case $\gamma_k \to x_k$.

Consequently, we obtain the following formula

$$\int_a^b \omega(x)\, f(x)\, Q_{m,\sigma}(x)\, d\alpha(x) =$$

$$= \int_a^b \omega(x)\, P_{m,\sigma}(x)\, Q_{m,\sigma}(x) [x, x_1, \ldots, x_m; f(t)]\, d\alpha(x),$$

where we have used the notations (14) and (15) and for shortness: $x_{k,\sigma} = x_k$.

Applying the mean-value formula to the integral from the second side of this equality, we get

$$\int_a^b \omega(x)\, f(x)\, Q_{m,\sigma}(x)\, d\alpha(x) = \tag{18}$$

$$= [\eta, x_1, \ldots, x_m; f(t)] \int_a^b \omega(x)\, P_{m,\sigma}(x)\, Q_{m,\sigma}(x)\, d\alpha(x).$$

Since $P_{m,\sigma}(x)\, Q_{m,\sigma}(x) = Q_{m,\sigma}(x^m + \ldots)$, if we use the orthogonality relations we may conclude that we have the following important mean-value formula

$$\int_a^b \omega(x)\, f(x)\, Q_{m,\sigma}(x)\, d\alpha(x) = \tag{19}$$

$$= [\eta, x_1, \ldots, x_m; f(t)] \int_a^b \omega(x)\, x^m Q_{m,\sigma}(x)\, d\alpha(x).$$

If $f \in C^m(a,b)$ and we have no preassigned nodes, by applying the mean-value formula to the divided difference involved, we get the following mean-value formula for the integrals, corresponding to the power-orthogonal polynomials:

$$\int_a^b f(x)\, Q_{m,\sigma}(x)\, d\alpha(x) = \frac{f^{(m)}(\xi)}{m!} \int_a^b x^m Q_{m,\sigma}(x)\, d\alpha(x), \quad a < \xi < b. \tag{20}$$

When $s_1 = \ldots = s_m = 0$ we are in the case of standard orthogonal polynomials and we arrive at the formula (1) of N. Cioranescu [3], while if $s_1 = \ldots = s_m = s$ we obtain the extension (2) of it, corresponding to the s-orthogonal polynomials $P_{m,s}$, in the sense of Turán, Ghizzetti, Ossicini, Rosati, Gori, Gautschi, Milovanović and their collaborators: [24], [6], [8], [5], [9], [10], [4], [14].

6. Now we want to find a formula which is similar to (19), by using the theory of T. Popoviciu [19] on functionals of simple form, having a certain degree of exactness. We consider the linear functional F defined by

$$F(f) = \int_a^b \omega(x)\, f(x)\, Q_{m,\sigma}(x)\, d\alpha(x). \tag{21}$$

If we take into account the equality (18), we can write $F(e_j) = 0$ $(j = 0, 1, \ldots, m-1)$, $F(e_m) > 0$. It follows that the functional F has the degree of exactness $m-1$.

Let us denote by $\varphi_m(x;t)$ the spline function

$$\varphi_m(x;t) = \left(\frac{x - t + |x - t|}{2}\right)^{m-1},$$

which has a continuous derivative of order $m-2$, and is non-concave of order $m-3$.

Since $F(\varphi_m) \geq 0$ on (a, b), by using a theorem of Popoviciu [19] we can conclude that there exist $m+1$ distinct points $t_1, \ldots, t_{m+1}$ in (a, b) such that

$$F(f) = F(e_m)[t_1, \ldots, t_{m+1}; f(t)].$$

According to (18) we have

$$F(e_m) = \int_a^b \omega(x)\, P_{m,\sigma}(x)\, Q_{m,\sigma}(x)\, d\alpha(x).$$

Consequently, we obtain the representation

$$\int_a^b \omega(x)\, f(x)\, Q_{m,\sigma}(x)\, d\alpha(x) =$$

$$= [t_1, \ldots, t_{m+1}; f(t)] \int_a^b \omega(x)\, P_{m,\sigma}(x)\, Q_{m,\sigma}(x)\, d\alpha(x).$$

Formula (18) has on it the advantage that the m nodes involved are known to be the zeros of the power orthogonal polynomial $P_{m,\sigma}(x)$.

7. In the case when we have no preassigned nodes, $s_k = s\ (k = 1, \ldots, m)$ and $d\alpha(x) = w(x)\,dx$, we can write: $Q_{m,\sigma}(x) = P_{m,s}^{2s+1}(x)$, where $P_{m,s}(x) = (x - x_1)\ldots(x - x_m)$ represents the s-orthogonal polynomial with respect to the weight function w on the interval (a, b).

Investigations on the s-orthogonal polynomials, minimizing the integral $\int_a^b w(x)\, P_{m,s}^{2s+2}(x)\, dx$, have been done by many mathematicians: D. Jackson [12], P. Turán [24], T. Popoviciu [17], L. Chakalov [2], A. Ghizzetti and A. Ossicini ([5], [6] and [7]), L. Gori [8], W. Gautschi [4], G. V. Milovanović [14], and others. In 1930 S. Bernstein [1] proved that, if $(a, b) = (-1, 1)$ and $w(x) = (1-x^2)^{-1/2}$, the extremal polynomial is just the Chebyshev polynomial of the first kind, for any nonnegative integer s.

There are other cases of weight functions (see [16] and [9]) for which the corresponding s-orthogonal polynomials are independent of the values of s.

Finally, we want to mention that, by using the mean-value formulae established in this paper, it will be important to investigate the generalized Fourier expansions of a function f in s or σ-orthogonal polynomials.

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Received September 15, 1997

Laura Gori
Department MeMoMatem
"La Sapienza" University
Via A. Scarpa, 10
00161 Roma, Italia

D. D. Stancu
Faculty of Mathematics and Computer Science
"Babeş-Bolyai" University
Str. Kogălniceanu, nr. 1
3400 Cluj-Napoca, Romania