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THE NUMERICAL SOLUTION OF FIRST ORDER VOLTERRA DELAY INTEGRO-DIFFERENTIAL EQUATIONS BY SPLINE FUNCTIONS

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1. INTRODUCTION

Consider the nonlinear first order Volterra delay integro-differential equation of the form:

$$
\begin{aligned}
& y'(x) = f\left(x, y(x), y(g(x)), \int_a^x K(x,t,y(t),y(g(t)))\,\mathrm{d}t\right), \quad a \le x \le b \\
(1) \qquad & y(a) = y_0 \\
& y(x) = \phi(x), \quad x \in [a^*, a),
\end{aligned}
$$

where f, g, K and ϕ are given functions and y is the unknown function to be found in the interval $[a,b]$. The equations of type (1) have found applications in many fields, such as control theory, physics, engineering and biology, therefore their numerical treatment is desired. Recently, many authors ([4] and [5]) have proposed different methods to approximate the solution of differential equation with deviating argument, and some authors ([2], [3], [6] and [7]) have proposed different methods to approximate the solution of integro-differential equation by means of spline functions.

In this paper, we use a polynomial spline function to find the approximate solution of problem (1). The method is a one-step method $o(h^r)$ in y and $o(h^{r+1+\alpha-1})$ in $y^{(i)}$ provided that $f \in \mathbb{C}^r([a,b]\times\mathbb{R}^3)$ and the modulus continuity of y' is $o(h^\alpha)$, where $r \in \mathbb{N}$, $i - 1(1)r+1$, and $0 < \alpha \le 1$. It is also shown that our method is A-stable.

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2. DESCRIPTION OF THE METHOD

Following [2], we shall write (1) in the following form:

$$(2)\qquad \begin{aligned} &y'(x)=f(x,y(x),y(g(x)),z(x)),\quad a\le x\le b\\ &z(x)=\int_a^x K(x,t,y(t),y(g(t)))\,dt\\ &y(a)=y_0\\ &y(x)=\phi(x),\quad x\in[a^*,a). \end{aligned}$$

The function g, called the delay function, is assumed to be continuous in the interval $[a^*,b]$ and to satisfy the inequality $a^*\le g(x)\le x, x\in[a,b]$. Suppose that $\phi\in\mathbb{C}^r[a^*,a]$, where $r\in\mathbb{N}$.

Assume that $f:[a,b]\times\mathbb{R}^3\to\mathbb{R}, (x,u,v,z)\to f(x,u,v,z)$ is defined and continuous together with its rth derivatives satisfying the Lipschitz conditions:

$$(3)\qquad |f^{(a)}(x,u_1,v_1,z_1)-f^{(a)}(x,u_2,v_2,z_2)|\le \le L_1\{|u_1-u_2|+|v_1-v_2|+|z_1-z_2|\}$$

and $\exists P<\min\left\{\frac{1}{L_1},\frac{1}{L_2}\right\}$ such that

$$(4)\qquad |v_1-v_2|\le P|f^{(q)}(x,u_1,v_1,z_1)-f^{(q)}(x,u_2,v_2,z_2)|$$

$\forall\,(x,u_1,v_1,z_1),(x,u_2,v_2,z_2)\in([a,b]\times\mathbb{R}^3)$ and $q=0(1)r$. Also, assume that $K:[a,b]\times[a,b]\times\mathbb{R}^2\to\mathbb{R}, (x,t,u,v)\to K(x,t,u,v)$ is a smooth, bounded function satisfying the Lipschitz conditions

$$(5)\qquad |K(x,t,u_1,v_1)-K(x,t,u_2,v_2)|\le L_2\{|u_1-u_2|+|v_1-v_2|\}$$

and

$$(6)\qquad |v_1-v_2|\le P|K(x,t,u_1,v_1)-K(x,t,u_2,v_2)|$$

$\forall\,(x,t,u_1,v_1),(x,t,u_2,v_2)\in([a,b]\times[a,b]\times\mathbb{R}^2)$. These conditions assure the existence of a unique solution of problem (2). We define the spline function approximating the solution y by S, where

$$(7)\qquad S(x)=\begin{cases}S_\Delta(x) & x\in[a,b]\\ \phi(x) & x\in[a^*,a].\end{cases}$$

Let Δ be the uniform partition of the interval $[a,b]$ defined by

$$a=x_0<\ldots x_k<\ldots<x_N=b,\ h=\frac{b-a}{N},\ x_{k+1}-x_k=h,\ k=0(1)\,N-1.$$

For any $x\in[x_k,x_{k+1}], k=0(1)\,N-1$, we define S_Δ by

$$(8)\qquad S_\Delta(x)=S_k(x)=S_{k-1}(x)+\sum_{i=0}^{r}\frac{M_k^{(i)}}{(i+1)!}(x-x_k)^{i-1}$$

$$(9)\qquad M_k^{(i)}=f^{(i)}\left[x_k,S_{k-1}(x_k),S_{k-1}(g(x_k)),\int_a^{x_k}K(x_k,t,S_{k-1}(t))\,dt\right]$$

$$i=0(1)\,r,$$

where

$$S_{-1}(x_0)=y_0,\quad S_{-1}(t)=y_0,$$
$$S_{-1}(g(x_0))=\phi(g(x_0)),\ S_{-1}(g(t))=\phi(g(t))$$

and $S_{k-1}(x_k)$ is the left-hand limit of $S_{k-1}(x)$ as $x\to x_k$ of the segment S_Δ defined on $[x_{k-1},x_k]$. Obviously, such $S_\Delta\in\mathbb{C}[a,b]$ exists and is unique.

3. ERROR ESTIMATION AND CONVERGENCE

In order to estimate the error, for all $x\in[x_k,x_{k+1}], k=0(1)\,N-1$, we suppose that the exact solution can be written by using Taylor's expansion in the following form

$$(10)\qquad y(x)=\sum_{i=0}^{r}\frac{y_k^{(i)}}{i!}(x-x_k)^i+\frac{y^{(r+1)}(\xi_k)}{(r+1)!}(x-x_k)^{r+1},$$

where $y_k^{(i)}=y^{(i)}(x_k), i=0(1)\,r, \xi_k\in[x_k,x_{k+1}]$ and the function $y^{(r+1)}$ has a modulus of continuity $\omega(y^{r+1},h)=\omega(h)=0(h^\alpha), 0<\alpha\le 1$. Moreover, we denote the estimated error of y at any point $x\in[a,b]$ by

$$(11)\qquad e(x):=|y(x)-S_\Delta(x)|;\quad e_k=|y_k-S_\Delta(x_k)|.$$

We need to use the following two lemmas.

LEMMA 3.1 [6]. *Let α and β be nonnegative real numbers, $\beta\ne 1$ and $\{A_i\}_{i=0}^{k}$ be a sequence satisfying $A_0\ge 0$ and $A_{i+1}\le\alpha+\beta A_i$ for $i=0(1)\,k$; then*

$$A_{k+1}\le\beta^{k+1}A_0+\frac{\alpha[\beta^{k+1}-1]}{\beta-1}.$$

By using direct calculus and induction, it is easy to prove the following.

LEMMA 3.2. *If*

$$y(x)=\sum_{i=0}^{n}\frac{y_k^{(i)}}{i!}(x-x_k)^i,$$

then for $p=0(1)\,n-1$

$$y^{(p)}(x)=\sum_{i=0}^{n-p}\frac{y_k^{(i+p)}}{i!}(x-x_k)^i.$$

LEMMA 3.3. *Let* $e(x)$ *be defined as in* (10); *then there exists some constant* b_2 *independent of* h *such that*

(12) $$e(x)\le b_2h^r\omega(h).$$

Proof. Using (9), (8), (3), (4), (5), (6) and (10), we get

(13) $$e(x)=|y(x)-S_\Delta(x)|\le e_k+\sum_{i=0}^{r-1}\frac{|y_k^{(i+1)}-M_k^{(i)}|h^{i+1}}{(i+1)!}+$$

$$+\frac{|y^{(r+1)}(\xi_k)-M_k^{(r)}|h^{r+1}}{(r+1)!}\le e_k+\sum_{i=0}^{r-1}\frac{T_ih^{i+1}}{(i+1)!}+\frac{T_rh^{r+1}}{(r+1)!},$$

where

$$T_i=|y_k^{(i+1)}-M_i^{(i)}|=|f^{(i)}[x_k,y_k,y(g(x_k)),\int_a^{x_k}K(x_k,t,y(t),y(g(t)))\,dt]-$$

$$-f^{(i)}[x_k,S_{k-1}(x_k),S_{k-1}(g(x_k)),\int_a^{x_k}K(x_k,t,S_{k-1}(t),S_{k-1}(g(t))\,dt]|\le$$

$$\le\frac{L_1}{1-PL_1}\left\{e_k+\int_a^{x_k}|K(x,t,y(t),y(g(t)))-K(x,t,S_{k-1}(t),S_{k-1}(g(t)))|\,dt\right\}\le$$

$$\le\frac{L_1}{1-PL_1}e_k+\frac{L_2}{1-PL_2}\int_a^{x_k}|y(t)-S_{k-1}(t)|\,dt.$$

But for $t\in[x_{k-1},x_k],e(t)=|y(t)-S_{k-1}(t)|\to e_k$ as $t\to x_k$, hence

(14) $$T_i\le b_0e_k,$$

where $b_0=\frac{L_1}{1-L_1P}\left[1+\frac{(b-a)L_2}{(1-L_2P)}\right]$ is a constant independent of h.

(15) $$T_i=|y^{(r+1)}(\xi_k)-M_k^{(r)}|\le|y^{(r+1)}(\xi_k)-y_k^{(r+1)}|+|y_k^{r+1}-M_k^{(r)}|\le$$

$$\le\omega(y^{(r+1)},h)+b_0e_k\le\omega(h)+b_0e_k.$$

Using (13) and (14) in (12), we get

(16) $$e(x)\le(1+hb_1)\,e_k+\frac{h^{r+1}}{(r+1)!}\omega(h),$$

where

$$b_1=b_0\sum_{i=0}^{r}\frac{1}{(i+1)!}$$

is a constant independent of h. The inequality (15) holds for any $x\in[a,b]$. Setting $x=x_{k+1}$, we get:

$$e_{k+1}\le(1+hb_1)\,e_k+\frac{h^{r+1}}{(r+1)!}\omega(h).$$

Using Lemma 3.1 and noting that $e_0=0$, we get

(17) $$e(x)\le b_2h^r\omega(h),$$

where

$$b_2=\frac{\exp[b_1(b-a)]}{b_1(r+1)!}$$

is a constant independent of h.

THEOREM 3.1. *Let* y *be the exact solution of problem* (1). *If* S_Δ *given by* (8) *is the approximate solution of the problem and* $f\in\mathbb{C}^r([a,b]\times\mathbb{R}^3)$, *then the inequalities*

$$|y(x)-S_\Delta(x)|\le b_3h^r\omega(h)$$

and

$$|y^{(q)}(x)-S_\Delta^{(q)}(x)|\le b_4h^{r+1-q}\omega(h)$$

hold for all $x\in[a,b],r\in\mathbb{N},q=1(1)\,r+1,b_3$ *and* b_4 *being constants independent of* h.

Proof. Using (9), (8), (3), (4), (5), (6), (10), (13) and (14), we can prove the theorem.

4. STABILITY OF THE METHOD

In order to study the stability of the method given by (8), we change S_Δ with W_Δ, where

$$W_\Delta(x) = W_k(x) = W_{k-1}(x_k) + \sum_{i=0}^{r} \frac{N_k^{(i)}}{(i+1)!}(x - x_k)^{i+1} \tag{18}$$

$$N_k^{(i)} = f^{(i)}\left[x_k, W_{k-1}(x_k), W_{k-1}(g(x_k)), \int_a^{x_k} K(x_k, t, W_{k-1}(t), W_{k-1}(g(t))\,dt\right],$$

$$i = 0(1)r,$$

where

$$W_{-1}(x_0) = y_0^*, W_{-1}(t) = y_0^*, W_{-1}(g(x_0)) = \phi(g(x_0)), W_{-1}(g(t)) = \phi(g(t))$$

and $W_{k-1}(x_k)$ is the left-hand limit $W_{k-1}^{\cdot}$ as $x \to x_k^{\cdot}$ of the segment W_Δ defined on $[x_{k-1}, x_k]$. Moreover, we use the following notation:

$$e^*(x) = |S_\Delta(x) - W_\Delta(x)|;\ e_k^* = |S_\Delta(x_k) - W_\Delta(x_k)|. \tag{19}$$

LEMMA 4.1. *Let $e^*(x)$ be defined as in (18), then there exists some constant b_5 independent of h such that*

$$e^*(x) \le b_5 e_0^* \tag{20}$$

and

$$e_0^* = |y_0 - y_0^*|.$$

Proof. Using (8), (17), (3), (4), (5), (6) and (18), the lemma follows directly.

THEOREM 4.1. *Let S_Δ given by (8) be the approximate of the exact solution of the problem (1) with initial condition y(a) = y_0 and let W_Δ, given by (17) be the approximate solution for the same problem with initial condition $y(a) = y_0^*$ and $f \in \mathbb{C}^r([a,b] \times \mathbb{R}^3)$. Then the following inequality $|S_\Delta^{(q)}(x) - W_\Delta^{(q)}(x)| \le b_6 e_0^*$ holds for all $x \in [a,b], r \in \mathbb{N}, q = 0(1)r$, $e_0^* = |y_0 - y_0^*|$, where b_6 is a constant independent of h.*

5. NUMERICAL EXAMPLES

The method is tested using the following two examples. We take the step size $h = 0.1$ and $r = 0,1$. In order to test the stability of the method, we make a

change in the starting value by adding 0.0001 to the initial value and solve the same problem.

Example 5.1. Consider the Volterra delay integro-differential equations

$$y'(x) = y\left(\frac{x}{2}\right) - e^{\frac{x}{2}} + \int_0^x y(t)\,dt + t, \quad 0 \le x \le 1$$

$$y(x) = e^x,\ x \le 0.$$

The exact solution is $y = e^x$.

In order to test the stability, we solve the same problem with $y(0) = 1.0001$.

x	First App. Spline F÷F		Absolute Error	Sec. App. Spline F÷F	|F – S|
0.1	r = 0	y = 1.1	5.2×10^{-3}	1.10011	1.1×10^{-4}
	r = 1	y = 1.105	1.7×10^{-4}	1.1051105	1.1×10^{-4}
	r = 0	y' = 1	1.1×10^{-1}	1.10001	1×10^{-4}
	r = 1	y' = 1.1	5.2×10^{-3}	1.100011	1.1×10^{-4}
	r = 1	y'' = 1	1.2×10^{-1}	1.10001	1×10^{-4}
0.2	r = 0	y = 1.21037289	1.1×10^{-2}	1.21049444	1.2×10^{-4}
	r = 1	y = 1.22103679	3.7×10^{-4}	1.221159258	1.2×10^{-4}
	r = 0	y' = 1.103728904	1.2×10^{-1}	1.103844404	1.2×10^{-4}
	r = 1	y' = 1.215582015	5.8×10^{-3}	1.215713956	1.3×10^{-4}
	r = 1	y'' = 1.104364452	1.2×10^{-1}	1.104527452	1.6×10^{-4}
0.3	r = 0	y = 1.331855798	1.8×10^{-2}	1.331990548	1.3×10^{-4}
	r = 1	y = 1.349248354	6.1×10^{-4}	1.349385388	1.4×10^{-4}
	r = 0	y' = 1.214829082	1.4×10^{-1}	1.214961082	1.3×10^{-4}
	r = 1	y' = 1.343042295	6.8×10^{-3}	1.343192738	1.5×10^{-4}
	r = 1	y'' = 1.21845092	1.3×10^{-1}	1.218628799	1.7×10^{-4}
0.4	r = 0	y = 1.465679961	2.6×10^{-2}	1.465829737	1.5×10^{-4}
	r = 1	y = 1.490953332	8.7×10^{-4}	1.491106523	1.7×10^{-4}
	r = 0	y' = 1.338241632	1.5×10^{-1}	1.338391892	1.5×10^{-4}
	r = 1	y' = 1.484475436	7.3×10^{-3}	1.484646955	1.7×10^{-4}
	r = 1	y'' = 1.348513129	1.4×10^{-1}	1.348712058	2×10^{-4}
0.5	r = 0	y = 1.61249824	3.6×10^{-2}	1.61266496	1.7×10^{-4}
	r = 1	y = 1.647557653	1.7×10^{-4}	1.647729195	1.7×10^{-4}
	r = 0	y' = 1.468182755	1.8×10^{-1}	1.4683552225	1.7×10^{-4}
	r = 1	y' = 1.640445355	8.3×10^{-3}	1.640639827	1.9×10^{-4}
	r = 1	y'' = 1.48804296	1.6×10^{-1}	1.488262122	2.2×10^{-4}

Example 5.2. Consider the Volterra delay integro-differential equations

$$y'(x) = y^2\left(\frac{x}{2}\right) + \int_0^x y^2\left(\frac{t}{2}\right)dt - e^x + 1, \quad 0 \le x \le 1$$

$$y(x) = e^x, \quad x \le 0.$$

The exact solution is $y = e^x$.

In order to test the stability, we solve the same problem with $y(0) = 1.0001$.

x	First App. Spline F÷F		Absolute Error	Sec. App. Spline F÷F	|F − S|
0.1	r = 0	y = 1.1	5.2×10^{-3}	1.1012001	1.2×10^{-4}
	r = 1	y = 1.105	1.7×10^{-4}	1.105122501	1.2×10^{-4}
	r = 0	y' = 1	1.1×10^{-1}	1.100020001	2×10^{-4}
	r = 1	y' = 1.1	5.2×10^{-3}	1.100250014	2.5×10^{-4}
	r = 1	y'' = 1	1.1×10^{-1}	1.10001	1×10^{-4}
0.2	r = 0	y = 1.210241241	1.1×10^{-2}	1.210386497	1.5×10^{-4}
	r = 1	y = 1.221031387	3.7×10^{-4}	1.1221182172	1.5×10^{-4}
	r = 0	y' = 1.102412415	1.2×10^{-1}	1.102664963	2.5×10^{-4}
	r = 1	y' = 1.215502281	5.9×10^{-3}	1.215814379	3.1×10^{-4}
	r = 1	y'' = 1.103768144	1.2×10^{-1}	1.104353474	5.8×10^{-4}
0.3	r = 0	y = 1.331167632	1.9×10^{-2}	1.331343916	1.8×10^{-4}
	r = 1	y = 1.349207656	6.5×10^{-4}	1.349535445	3.3×10^{-4}
	r = 0	y' = 1.209263909	1.4×10^{-1}	1.209574194	3.1×10^{-4}
	r = 1	y' = 1.342518799	7.3×10^{-3}	1.342904426	3.9×10^{-4}
	r = 1	y'' = 1.215122242	1.35×10^{-1}	1.215804031	6.8×10^{-4}
0.4	r = 0	y = 1.464184284	2.8×10^{-2}	1.464367915	1.8×10^{-4}
	r = 1	y = 1.49087508	9.5×10^{-4}	1.491246057	3.7×10^{-4}
	r = 0	y' = 1.330166519	1.6×10^{-1}	1.33023999	0.7×10^{-4}
	r = 1	y' = 1.484044368	7.8×10^{-3}	1.484516304	4.7×10^{-4}
	r = 1	y'' = 1.347401697	1.4×10^{-1}	1.348203758	8×10^{-4}
0.5	r = 0	y = 1.609129113	4×10^{-2}	1.609358297	2.3×10^{-4}
	r = 1	y = 1.647382742	1.3×10^{-4}	1.647806146	4.2×10^{-4}
	r = 0	y' = 1.449448288	2×10^{-1}	1.44990382	4.6×10^{-4}
	r = 1	y' = 1.639239589	9.5×10^{-3}	1.63981049	5.7×10^{-4}
	r = 1	y'' = 1.483259385	1.7×10^{-1}	1.484192044	9.3×10^{-4}

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