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GENERALIZED STANCU-PÓLYA CURVES*

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1. INTRODUCTION

Bézier curves are widely known as one of the basic tools in Computer Aided Geometric Design.

The Bézier curve corresponding to a control polygon $\mathbf{P} = [\mathbf{P}_0 \ldots \mathbf{P}_m]^T$, $\mathbf{P}_j \in \mathbf{R}^2$, is an m-th degree polynomial curve whose parametric equation is the following

$$(1.1) \qquad \mathbf{B}_m[\mathbf{P}](t) = \sum_{j=0}^{m} p_{m,j}(t)\,\mathbf{P}_j, \quad t \in [0,1],$$

its blending functions being the Bernstein basis polynomials

$$(1.2) \qquad p_{m,j}(t) = \binom{m}{j} t^j (1-t)^{m-j}.$$

Then each component of (1.1) can be regarded as being obtained by means of the Bernstein operator B_m from any continuous function f such that $f(i/m) = P_i^c$, $i = 0, \ldots, m$, through the relation

$$B_m^c(t) = B_m(f;t), \quad t \in [0,1],$$

where either $c = x$ or $c = y$.

Piecewise Bézier curves, such as B-spline curves, also widely used in CAGD, are usually considered as the natural generalization of Bézier curves ([1], [3] and [4]). But polynomial generalizations can be introduced, too, namely polynomial curves that depend on some parameters and, for special values of these, reduce to Bézier curves.

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Pólya curves considered in [1], [2], [6] are such curves. Their parametric equation is

$$S_m^\alpha[\mathbf{P}](t)=\sum_{j=0}^{m} w_{m,j}^\alpha(t)\,\mathbf{P}_j,\quad t\in[0,1];\quad \alpha\in\Re_m, \tag{1.3}$$

where the blending functions are the Stancu basis polynomials [13]

$$w_{m,j}^\alpha(t)=\binom{m}{j}\frac{t^{[j,-\alpha]}(1-t)^{[m-j,-\alpha]}}{1^{[m,-\alpha]}}, \tag{1.4}$$

with

$$t^{[h,\alpha]}=\begin{cases}1, & h=0\\ t(t-\alpha)\dots(t-(h-1)\,\alpha), & h\ge 1,\end{cases}$$

and $\Re_m=\Re-\left\{-\frac{1}{k}\right\}_{k=1}^{m-1}$.

Denoting by S_m^α the Stancu operator [13], the component c of the Pólya curve (1.3) ($c=x$ *or* y) is given by

$$S_m^c(t)=S_m^\alpha(f;t),\ t\in[0,1],$$

where, again, f is any continuous function such that $f(i/m)=P_i^c$, $i=0,\dots,m$.

Stancu polynomials specialize to Bernstein polynomials for $\alpha=0$ and to Lagrange polynomials based on equally spaced knots in [0, 1] for $\alpha=-1/m$, therefore (1.3) reduces to (1.1) for $\alpha=0$, and to the equation of the Lagrange curve interpolating the control polygon $\mathbf{P}$ for $\alpha=-1/m$. In this sense, when $\alpha\in\left[-\frac{1}{m},0\right]$, Pólya curves link Bézier to Lagrange curves [5].

In the following, we shall denote the interval $\left[-\frac{1}{m},0\right]$ by J_m and shall assume $\alpha\in J_m$ unless it is stated otherwise.

A different polynomial generalization of Bézier curves is introduced in [11] and [12] by means of the generalized Bernstein operator $B_{m,k}$ ([8] and [9]). The latter is defined as

$$B_{m,k}=I-(I-B_m)^k,\quad k\in N.$$

Curves in this new scheme can be defined componentwise by

$$B_{m,k}^c(t)=B_{m,k}(f;t),\quad t\in[0,1];\ f(i/m)=P_i^c,\ i=0,\dots,m$$

for $c=x$ and $c=y$. We refer to them as GB curves (Generalized Bézier). Their parametric equation is

$$\mathbf{B}_{m,k}[\mathbf{P}](t)=\sum_{j=0}^{m} p_{m,j}^{(k)}(t)\,\mathbf{P}_j,\quad t\in[0,1],\ k\in N, \tag{1.5}$$

and their blending functions are

$$p_{m,j}^{(k)}(t)=\sum_{i=1}^{k}\binom{k}{i}(-1)^{i-1}B_m^{i-1}(p_{m,j};t), \tag{1.6}$$

where B_m^j denotes the j-th iterate of the Bernstein operator, i.e., $B_m^j=B_m^{j-1}(B_m)$, $B_m^0=I$.

GB curves also link Bézier to Lagrange curves, since the polynomial $B_{m,k}(f)$ reduces to $B_m(f)$ if $k=1$, and it tends to the Lagrange polynomial interpolating f on the equally spaced knots as $k\to\infty$ [8].

Furthermore, as proved in [11], equation (1.5) can be put in the following more convenient form

$$\mathbf{B}_{m,k}[\mathbf{P}](t)=\sum_{j=0}^{m} p_{m,j}^{(k)}(t)\,\mathbf{T}_j,\quad t\in[0,1], \tag{1.7}$$

and the Bézier polygon $\mathbf{T}$ of the GB curve is related to its control polygon $\mathbf{P}$ by means of a simple vector relation, namely

$$\mathbf{T}=[\mathbf{T}_0,\dots,\mathbf{T}_m]^T=C_{m,k}\mathbf{P}, \tag{1.8}$$

where $C_{m,k}$ is a centrosymmetric matrix that can be computed from the collocation values of the Bernstein basis at the equispaced points $\{i/m\}_{i=0}^m$. Actually, $C_{m,k}$ is the transformation matrix between the Bernstein basis and the $\{p_{m,j}^{(k)}(t)\}_{j=0}^m$ basis in the space Π_m of polynomials of degree at most m.

The main difference between these two generalizations of the Bézier scheme lies in the range where their respective parameters take values. In fact, Pólya curves permit the passage from a Bézier curve to a Lagrange curve continuously changing its shape by means of the real valued parameter α, while GB curves' discrete parameter $k\in N$ does not allow such continuous *morphing*. In addition, Pólya curve reduces exactly to the Lagrange curve for $\alpha=-1/m$, while the Lagrange case is just a limiting case in the GB scheme.

However, Pólya curves also have drawbacks. If m is large, they exhibit instability phenomena when α approaches $-1/m$, [7]. Furthermore, in such case they do not provide as good a mimic reproduction of the control polygon and as high a rate of convergence as the GB curve scheme. Finally, the GB scheme is computationally more convenient. A complete comparison of the two curve schemes was performed in [7].

In order to overcome the above mentioned drawbacks and, at the same time, retain the fair properties of these schemes, we introduce here a wider class of curves which includes both as special cases. Curves in this new class, to which we shall refer as GSP curves (short for Generalized Stancu-Pólya curves), are obtained as a by-product of the generalized Stancu operator $S^{\alpha}_{m,k}$ introduced in [10]. This is defined as

$$S^{\alpha}_{m,k} = I - (I - S^{\alpha}_{m})^{k}, \quad k \in N, \quad \alpha \in \Re_m. \tag{1.9}$$

and can be expressed as a linear combination of iterates of S^{α}_{m}.

The GSP-curve corresponding to the control polygon $\mathbf{P} = [\mathbf{P}_0 \ldots \mathbf{P}_m]^T$, $\mathbf{P}_j \in \mathbf{R}^2$, is defined componentwise by

$$S^{\alpha}_{m,k}(t) = S^{\alpha}_{m,k}(f;t), \quad t \in [0,1]; \quad f(i/m) = P^{c}_{i} \; i = 0, \ldots, m,$$

for $c = x$ and $c = y$.

If $k = 1$, then $S^{\alpha}_{m,k}$ reduces to S^{α}_{m}, while if $\alpha = 0$ it reduces to $B_{m,k}$, therefore the class of GSP curves is an extension of both the class of Pólya curves and the class of GB curves. As we shall see, it is a proper extension of both, in the sense that, for a fixed control polygon, there are GSP curves that cannot be obtained as Pólya curves, nor as GB curves, from the same control polygon. Obviously, Lagrange curves are also included, as a special case, in the class of GSP curves, since the Lagrange polynomial can be obtained from (1.9) either for $\alpha \geq 0$ and $k \to \infty$ or for $k \in N$ and $\alpha = -1/m$. In this sense, GSP curves also link continuously Bézier to Lagrange curves.

The GSP curve scheme, therefore, offers a useful unifying context including polynomial schemes, such as Bézier, Pólya and Lagrange, that are the basic ones in CAGD. Furthermore, in this wider context, where two shape parameters are available, careful combination of these may allow us, for example, to overcome the instability problems sometimes presented by Pólya curves, choosing α far from the critical value $-1/m$, and increasing k freely.

The plan of this paper is the following: in the next section we prove that GSP is actually a new curve scheme, in the sense that it really generalizes Pólya and GB schemes. In section 3 we derive a vector form of the curve equation that is a useful tool both to investigate geometrical properties of the new class of curves and to construct efficient rendering algorithms. We prove some geometrical

properties in Section 4, and present efficient evaluation algorithms in Section 5. Finally, in Section 6, we give graphical examples, showing cases in which careful adjustment of the two parameters permits us to obtain GSP curves that mimic very closely the control polygon, avoiding the instability problems related to Pólya curves.

2. PRELIMINARY RESULTS

In this section we show that

$$\{S^{\alpha}_{m}(t)\}_{\alpha \in J_m} \subset \{S^{\alpha}_{m,k}(t)\}_{\alpha \in J_m, k \in N}. \tag{2.1}$$

Since, obviously

$$\forall \alpha \in \Re_m \quad S^{\alpha}_{m}(f) = S^{\alpha}_{m,1}(f),$$

it suffices to prove that $\alpha \in J_m$ and $k \in N$ exist such that

$$S^{\alpha}_{m,k}(f) \notin \{S^{\beta}_{m}(f)\}_{\beta \in J_m}. \tag{2.2}$$

Indeed, we shall prove that, for any fixed $k > 1$, A_k, a finite subset of $\Re$, can be determined, such that (2.2) holds $\forall \alpha \in (\Re_m - A_k) \cap J_m$. This means that there are infinitely many values of α such that the corresponding GSP curve *is not* a Pólya curve for the same control polygon.

In order to prove this result, we shall resort to the same technique, based on operator's eigenfunctions, as used in [7]. For this, first we need to derive some properties of the spectra of S^{α}_{m} and of $S^{\alpha}_{m,k}$.

By definition, $\nu(\alpha)$ is an eigenvalue of S^{α}_{m}, and $q(x)$ is a corresponding eigenfunction, if $S^{\alpha}_{m}(q;x) = \nu(\alpha)\, q(x)$. Since S^{α}_{m} transforms degree i polynomials into degree i polynomials if $i \leq m$ [13], its eigenfunctions must be polynomials of increasing degree, $\{q_i(x)\}^{m}_{i=0}$. Therefore we are able to state the following

LEMMA 1. *The eigenvalues* $\{\nu_i(\alpha)\}^{m}_{i=0}$ *of the Stancu operator* S^{α}_{m} *are*

$$\nu_i(\alpha) = \frac{1^{[i,1/m]}}{1^{[i,-\alpha]}} = \frac{\lambda_i}{1^{[i,-\alpha]}}, \quad i = 0, \ldots, m, \tag{2.3}$$

where λ_i $(i = 0, \ldots, m)$ *denote the eigenvalues of the Bernstein operator. The eigenfunction corresponding to* $\nu_i(\alpha)$ *is a polynomial of degree* i.

Proof. The method of undetermined coefficients can be applied to determine both the eigenvalues and the coefficients of the eigenfunctions of S_m^α, starting from the relation $S_m^\alpha(q_i) = v_i(\alpha)\, q_i$, in a similar way as done, for example, in [14]. □

Remark 1. The first four eigenfunctions of S_m^α are :

$$q_0(x) = 1,\ q_1(x) = x,\ q_2(x) = x^2 - x,\ q_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

We recall that these are also the first four eigenfunctions of B_m and of $B_{m,k}$, $\forall k$ [7].

LEMMA 2. *The eigenvalues $\{\sigma_i(\alpha, k)\}_{i=0}^m$ of the operator $S_{m,k}^\alpha$ are given by*

$$\sigma_i(\alpha, k) = 1 - (1 - v_i(\alpha))^k,\quad i = 0, \ldots, m, \tag{2.4}$$

where $v_i(\alpha)$ denotes the i-th eigenvalue of the Stancu operator. The corresponding eigenfunctions are those of S_m^α.

Proof. It is straightforward from (1.9) and from the fact that S_m^α is a linear operator. □

Remark 2. According to (2.4), if $\alpha \in J_m, k \in N$, then $0 < v_i(\alpha) \le \sigma_i(\alpha, k) \le 1$, $i = 0, \ldots, m$.

Now we are able to prove the following

THEOREM 1. *If $f \in \Pi_3$, $k > 1$, $J_m = [-1/m, 0]$, there are infinitely many values $\alpha \in J_m$ such that*

$$S_{m,k}^\alpha(f) \ne S_m^\beta(f) \quad \forall \beta \in J_m. \tag{2.5}$$

Proof. Since $\{q_i\}_{i=0}^3$ is a basis for Π_3, exist $\{a_i\}_{i=0}^3, a_i \in \Re$ such that $f(x) = a_0 q_0(x) + a_1 q_1(x) + a_2 q_2(x) + a_3 q_3(x)$. Therefore, because of the linearity of the operators involved, the relation

$$S_{m,k}^\alpha(f) = S_m^\beta(f) \tag{2.6}$$

is equivalent to

$$\sum_{i=0}^3 a_i \sigma_i(\alpha, k)\, q_i = \sum_{i=0}^3 a_i v_i(\beta)\, q_i. \tag{2.7}$$

According to (2.3) and (2.4), (2.7) reduces to the following system of two algebraic equations on β, in which k is fixed, while α is a parameter

$$\begin{cases} a_2 v_2(\beta) = a_2(1 - [1 - v_2(\alpha)]^k) \\ a_3 v_3(\beta) = a_3(1 - [1 - v_3(\alpha)]^k). \end{cases} \tag{2.8}$$

Relations (2.8) trivially hold in the case $a_2 = a_3 = 0$, that is, if f is a linear function. Actually, in this case, $S_{m,k}^\alpha(f) = S_m^\alpha(f), \forall k$. Indeed, this fact is well known [10].

Suppose, now, that we are in the case $a_2 \ne 0, a_3 = 0$. Then the system (2.8) reduces to the single equation on β

$$v_2(\beta) = (1 - [1 - v_2(\alpha)]^k),$$

which yields the solution

$$\beta = \frac{\lambda_2}{\sigma_2(\alpha, k)} - 1. \tag{2.9}$$

This is always an acceptable solution, since if $\alpha \in J_m$ (2.9) implies that also $\beta \in J_m$. Therefore, if f is a quadratic polynomial, for any $\alpha \in J_m$ there is one and only one value of β such that (2.6) holds, and it is given by (2.9).

Let us now examine the case $a_2 = 0, a_3 \ne 0$. In this case, (2.8) reduces to the quadratic equation on β

$$2\beta^2 + 3\beta + \left[1 - \frac{\lambda_3}{\sigma_3(\alpha, k)}\right] = 0. \tag{2.10}$$

Since

$$\Delta = 1 + \frac{8\lambda_3}{\sigma_3(a, k)} > 0,$$

equation (2.10) admits real solutions $\beta_1 < \beta_2$, namely

$$\beta_1 = -\frac{3}{4} - \frac{\Delta}{4}, \tag{2.11}$$

$$\beta_2 = -\frac{3}{4} + \frac{\Delta}{4}.$$

If $\alpha \in J_m$, then $\beta_1 \le -1/m$ is not an acceptable solution. β_2, on the contrary, is in J_m, so it is acceptable.

Therefore, if f is the polynomial

$$f(x)=a_0q_0(x)+a_1q_1(x)+a_3q_3(x),$$

for any $\alpha\in J_m$, one and only one value $\beta(\alpha,k)$ can be determined according to equation (2.10), such that (2.6) is satisfied, and it is given by (2.11).

Finally, we must examine the case $a_2\neq 0, a_3\neq 0$. In order to be a solution of the system (2.8), β must obey (2.9) and (2.11) at the same time. This can only happen if the parameter α is a solution of

$$\frac{\lambda_2}{\sigma_2(\alpha,k)}-1=\frac{1}{4}\left(\sqrt{1+8\frac{\lambda_3}{\sigma_3(\alpha,k)}}-3\right). \tag{2.12}$$

Indeed, (2.12) is a $4k$ degree equation on α. Therefore, for any fixed $k>1$ there are at most $4k$ different values of α such that the corresponding system (2.8) admits a feasible solution. Let us denote the set of these values by $A_k=\{\alpha_1,\alpha_2,\ldots,\alpha_{4k}\}$, then, for any fixed $k>1$, A_k is a finite set and therefore $J_m\cap(\Re_m-A_k)$ is infinite. Furthermore, under the theorem's assumptions, $\forall\alpha\in J_m\cap(\Re_m-A_k)$ (2.5) holds. □

3. VECTOR FORM OF THE $S^{\alpha}_{m,k}(f)$ POLYNOMIAL

In order to keep our notation not too cumbersome, in the following we shall drop the index α from the symbols denoting the operators or the relative blending functions, the dependence on the parameter α being understood anyway. We shall write $S_{m,k}$ for $S^{\alpha}_{m,k}$, $w_{m,i}(t)$ for $w^{\alpha}_{m,i}(t)$, and so on.

It follows from (1.9) that

$$S_{m,k}=I-(I-S_m)^k=\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}S^j_m, \tag{3.1}$$

where $S^i_m=S_m(S^{i-1}_m)$, $S^0_m=I$.

Therefore, for any continuous function f, the polynomial $S_{m,k}(f)$ is given by

$$S_{m,k}(f;t)=\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}S^j_m(f;t),$$

which easily yields the expression

$$S_{m,k}(f;t)=\sum_{i=0}^{m}w^{(k)}_{m,i}(t)\,f\left(\frac{i}{m}\right),\quad 0\le t\le 1, \tag{3.2}$$

in which

$$w^{(k)}_{m,i}(t)=\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}S^{j-1}_m(w_{m,i};t). \tag{3.3}$$

Now we derive a different, more handy form for the polynomial $S_{m,k}(f)$, namely a vector form.

THEOREM 2. *For any function* $f\in C^0([0,1])$, *the polynomial* $S_{m,k}(f)$ *admits the following representation*

$$S_{m,k}(f;t)=\mathbf{w}_m(t)^T C^{\alpha}_{m,k}\mathbf{f}_m \tag{3.4}$$

or, equivalently, the blending functions (3.3) *satisfy*

$$\mathbf{w}^{(k)}_m(t)^T=\mathbf{w}_m(t)^T C^{\alpha}_{m,k},$$

where

$$\mathbf{w}^{(k)}_m(t)=[w^{(k)}_{m,0}(t),\ldots,w^{(k)}_{m,m}(t)]^T,\quad \mathbf{w}_m(t)=[w_{m,0}(t),\ldots,w_{m,m}(t)]^T$$

$$\mathbf{f}_m=[f_0,f_1,\ldots,f_m]^T\quad f_i=f(i/m)$$

$$C^{\alpha}_{m,k}=[I_m+(I_m-A_m(\alpha))+(I_m-A_m(\alpha))^2+\ldots(I_m-A_m(\alpha))^{k-1}], \tag{3.5}$$

$$k\in N,$$

and $A_m(\alpha)\in R^{(m+1,m+1)}$ *is the collocation matrix of the Stancu basis polynomials at the points* $\{i/m\}^m_{i=0}$, *i.e.*,

$$A_m(\alpha)=(a_{i,j})_{i=0,\ldots,m,\,j=0,\ldots,m},\quad a_{i,j}=w_{m,j}\left(\frac{i}{m}\right). \tag{3.6}$$

Proof. According to (3.5), an equivalent form for (3.4) is

$$S_{m,k}(f;t)=\mathbf{w}_m(t)^T\left[\sum_{i=1}^{k}\binom{k}{i}(-1)^{i-1}A^{i-1}_m(\alpha)\right]\mathbf{f}_m.$$

Therefore, to prove (3.4) it suffices to prove that

$$S^i_m(f;t)=\mathbf{w}_m(t)^T A^{i-1}_m(\alpha)\mathbf{f}_m\quad i=1,\ldots,k. \tag{3.7}$$

We can do this by induction over i. Since (3.7) trivially holds if $i=1$, we assume that it holds for some i and prove that it does for $i+1$, too. Denoting by $A_m^i(\alpha)$ the i-th power of $A_m(\alpha)$, and by $a_{jk}^{(i)}$ its elements, we have

$$S_m^{i+1}(f;t) = S_m(S_m^i(f);t) = \sum_{k=0}^{m} w_{m,k}(t)\, S_m^i\left(f;\frac{k}{m}\right) =$$

$$= \sum_{k=0}^{m} w_{m,k}(t) \sum_{l=0}^{m} f\left(\frac{l}{m}\right) \sum_{j=0}^{m} w_{m,j}\left(\frac{k}{m}\right) a_{jl}^{(i-1)} =$$

$$= \sum_{l=0}^{m} f\left(\frac{l}{m}\right) \sum_{k=0}^{m} w_{m,k}(t) \sum_{j=0}^{m} a_{kj} a_{jl}^{(i-1)} =$$

$$= \sum_{l=0}^{m} f\left(\frac{l}{m}\right) \sum_{k=0}^{m} w_{m,k}(t)\, a_{kl}^{(i)}\, \mathbf{w}_m(t)^T A_m^i(\alpha)\, \mathbf{f}_m,$$

which proves our assertion. □

We want to point out explicitly that, for $\alpha = 0$, (3.4) yields (1.7) in vector form (see[11]). Moreover, we observe that $C_{m,k}^{\alpha}$ is a centrosymmetric matrix. In fact, denoting its elements by $\{c_{i,j}\}_{i,j=0,m}$, the relation $c_{i,j} = c_{m-i,m-j}$ holds, $\forall\, i,j = 0,m$. Furthermore, $\forall\, i = 0,m$, $\sum_{j=0}^{m} c_{ij} = 1$ holds.

As regards the eigenvalues of $C_{m,k}^{\alpha}$, we have the following

THEOREM 3. *The eigenvalues of the matrix $C_{m,k}^{\alpha}$ are*

$$\mu_i(\alpha,k) = \frac{[1-(1-v_i(\alpha))^k]}{v_i(\alpha)} = \frac{\sigma_i(\alpha,k)}{v_i(\alpha)}, \quad i = 0,\ldots,m, \tag{3.8}$$

where $v_i(\alpha)$ and $\sigma_i(\alpha,k)$ denote the eigenvalues of S_m and of $S_{m,k}$, respectively.

Proof. We prove that the eigenvalues of the matrix $A_m(\alpha)$ are $\{v_i(\alpha)\}_{i=0}^m$, then the assertion follows from (3.5). Let $q_i(x)$ denote the eigenfunction of S_m corresponding to $v_i(\alpha)$, so that

$$S_m(q_i) = v_i(\alpha)\, q_i, \quad m \ge i, \quad q_i \in \Pi_i. \tag{3.9}$$

Setting

$$\mathbf{w}_m(t) = [w_{m,0}(t),\ldots,w_{m,m}(t)]^T,$$

$$\mathbf{q}_m(t) = [q_0(t),\ldots,q_m(t)]^T,$$

$$\gamma_i = [q_i(0), q_i(1/m),\ldots,q_i(1)]^T$$

$$\Gamma = [\gamma_0,\gamma_1,\ldots,\gamma_m], \quad \Lambda = diag\,[v_0(\alpha), v_1(\alpha),\ldots,v_m(\alpha)],$$

(3.9) can be rewritten as

$$\mathbf{w}_m(t)^T \gamma_i = q_i(t)\, v_i(\alpha), \quad i = 0,\ldots,m,$$

that is,

$$\mathbf{w}_m(t)^T \Gamma = \mathbf{q}_m(t)^T \Lambda. \tag{3.10}$$

Grouping together the vector relations obtained setting $t = 0, 1/m,\ldots,1$ in (3.10), yields the matrix relation

$$A(\alpha)\,\Gamma = \Gamma\Lambda.$$

Γ is nonsingular, since $\{q_i\}_{i=0}^m$ is a basis for the space Π_m, therefore

$$A(\alpha)\,\Gamma = \Gamma\Lambda\Gamma^{-1}, \tag{3.11}$$

by which $A(\alpha)$ and Λ have the same eigenvalues $\{v_i(\alpha)\}_{i=0}^m$, and the assertion follows. □

Remark 4. It follows from Theorem 3 that, for $\alpha \in J_m$, $C_{m,k}^{\alpha}$ is nonsingular and, therefore, $\{w_{m,j}^{(k)}(t)\}_{j=0}^m$ is a basis for the space of polynomials Π_m. We deduce from Theorem 2 that $C_{m,k}^{\alpha}$ is the transformation matrix between the two bases $\{w_{m,j}(t)\}_{j=0}^m$ and $\{w_{m,j}^{(k)}(t)\}_{j=0}^m$.

4. DEFINITION AND PROPERTIES OF GSP CURVES

Given a control polygon $\mathbf{P} = [\mathbf{P}_0,\ldots,\mathbf{P}_m]^T, \mathbf{P}_j \in \Re^2$, we define the related Generalized Stancu-Pólya curve (or GSP curve) as the curve of equation

$$S_{m,k}^{\alpha}[\mathbf{P}](t) = \sum_{j=0}^{m} w_{m,j}^{(k)}(t)\, \mathbf{P}_j, \quad t \in [0,1]; \quad \alpha \in \mathbf{R}_m, \quad k \in N, \tag{4.1}$$

where the blending functions $w_{m,j}^{(k)}$ are given by (3.3).

In vector form:

$$\mathbf{S}_{m,k}[\mathbf{P}](t) = \mathbf{w}_m^{(k)}(t)^T \mathbf{P}, \quad t \in [0,1]. \tag{4.2}$$

This is equivalent to the following componentwise definition:

(4.3) $S^c_{m,k}(t) = S_{m,k}(f;t),\ t \in [0,1];\ \ f(i/m) = P^c_i,\ i = 0,\dots,m;\ \ c = x, y.$

Special cases of GSP curves, corresponding to special choices of the parameters α and k, are:

1) Lagrange curve for $\alpha = -1/m$ and $\forall k \in N$;
2) Lagrange curve for $\alpha \in \Re_m$ and $k \to \infty$;
3) Pólya curves for $\alpha \in \Re_m$ and $k = 1$;
4) GB curves for $\alpha = 0$ and $k \in N$;
5) Bézier curve for $\alpha = 0$ and $k = 1$.

Now we prove some geometrical properties of GSP curves.

Pólya form of GSP curve. Setting

$$\mathbf{T}^\alpha = [\mathbf{T}_0,\dots,\mathbf{T}_m]^T = C^\alpha_{m,k}\mathbf{P}, \tag{4.4}$$

according to Theorem 2, equation (4.2) takes the form

$$\mathbf{S}_{m,k}[\mathbf{P}](t) = \mathbf{w}_m(t)^T\,\mathbf{T}^\alpha, \quad t \in [0,1]. \tag{4.5}$$

This new form of the equation of the curve allows us to observe that GSP curve (4.1) can be looked upon as being itself the Pólya curve of a different control polygon, namely $\mathbf{T}^\alpha$. Therefore, all known algorithms for Pólya curves can be used for GSP curves, provided these are regarded as Pólya curves with respect to the new control polygon $\mathbf{T}^\alpha$.

Bézier form of GSP curve. Furthermore, since

$$\mathbf{w}_m(x)^T = \mathbf{p}_m(x)^T[A_m^{-1}A_m(\alpha)],$$

where $\mathbf{p}_m(x) = [p_{m,0}(t),\dots,p_{m,m}(t)]^T$, A_m denotes the collocation matrix of the Bernstein basis at the equispaced knots $\{i/m\}_{i=0}^m$, and $A_m(\alpha)$ denotes the collocation matrix of the Stancu basis given by (3.6), it follows from (4.4) and (4.5) that

$$\mathbf{S}_{m,k}[\mathbf{P}](t) = \mathbf{p}_m(t)^T\,\mathbf{D}^\alpha, \quad t \in [0,1], \tag{4.6}$$

where

$$\mathbf{D}^\alpha = D^\alpha_{m,k}\mathbf{P}$$

and

$$D^\alpha_{m,k} = [A_m^{-1}A_m(\alpha)]\,C^\alpha_{m,k}.$$

Relation (4.6) is analogous to relation (1.8) in the sense that it gives some insight of the geometrical nature of GSP curves, through knowledge of its Bézier polygon.

Some geometrical properties of GSP curves can also be deduced directly from the corresponding properties of their blending functions (3.3), following the scheme given in [6].

The rest of this section is devoted to this. But first we want to point out explicitly that, since GSP curves do not enjoy the convex hull property, that is, they are not in the convex hull of their control polygon, just as Pólya curves with $\alpha \in J_m$ are not, relation (4.6) is a tool of great practical value, as it provides a means for determining a convex region in which the curve fully lies, namely the convex hull of its Bézier polygon $\mathbf{D}^\alpha$.

We must also notice that, since they do not enjoy the convex hull property, GSP curves are not variation-diminishing, either. But they are endowed with the following geometrical properties:

Well defined. From $\sum_{j=0}^m c_{ij} = 1,\ \forall i = 0,1,\dots,m$ and $\sum_{j=0}^m w_{m,j}(t) = 1$ it easily follows that $\sum_{j=0}^m w^{(k)}_{m,j}(t) = 1$, which proves that GSP curves are well defined (see [6]).

Smoothness. It is trivially true since GSP are polynomial curves.

Endpoint Interpolation. This property holds if the following relations hold

$$\mathbf{S}_{m,k}[\mathbf{P}_0,\dots,\mathbf{P}_m](0) = \mathbf{P}_0, \quad \mathbf{S}_{m,k}[\mathbf{P}_0,\dots,\mathbf{P}_m](0) = \mathbf{P}_m.$$

And, indeed,

$$\mathbf{S}_{m,k}[\mathbf{P}](0) = \sum_{j=0}^m w^{(k)}_{m,j}(0)\,\mathbf{P}_j = \sum_{j=0}^m \mathbf{P}_j \sum_{i=0}^m w_{m,i}(0)\,c_{i,j} = \sum_{j=0}^m \mathbf{P}_j c_{0,j} = \mathbf{P}_0,$$

since $c_{0,j} = 0,\ j \neq 0$. The second relation can be proved similarly.

Symmetry. Symmetry property holds if

$$w^{(k)}_{m,j}(t) = w^{(k)}_{m,m-j}(1-t), \quad t \in [0,1], \quad j = 0, m.$$

Since $C^\alpha_{m,k}$ is centrosymmetric and $w_{m,i}(t) = w_{m,m-i}(1-t)$, then

$$w^{(k)}_{m,j}(t) = \sum_{i=0}^m w_{m,i}(t)\,c_{i,j} =$$

$$= \sum_{i=0}^m w_{m,m-i}(1-t)\,c_{m-i,m-j} = w^{(k)}_{m,m-j}(1-t), \quad j = 0,\dots,m.$$

Reproduction of points and lines. Since the curve is well defined, reproduction of points is assured. As regards the reproduction of lines, it is sufficient to prove that

$$\sum_{j=0}^{m} j w_{m,j}^{(k)}(t) = mt. \tag{4.7}$$

Since [10]

$$S_{m,k}(e_1;t) = e_1(t), \quad e_1(t) = t,$$

(4.7) follows.

Nondegeneracy. The curve is well defined, and its blending functions (3.3) are a basis for the space of polynomials of degree m (see *Remark* 4), so this property holds, according to [6].

5. RENDERING ALGORITHMS

Efficient rendering algorithms for GSP curves can be obtained from equations (4.5) and (4.6). These do not require evaluation of the blending functions $w_{m,j}^{(k)}$ at each value of the parameter t as resorting, instead, to (4.1) would require. Moreover, the use of an expression of type (1.6) may cause computational instability, while these algorithms do not. For effects on the evaluation of GB curves see, for instance [12].

According to (4.5), $S_{m,k}[\mathbf{P}]$ can be regarded as the Pólya curve of the new control polygon $\mathbf{T}^{\alpha}$; this can be constructed first, from (4.4), and then the points of the curve can be rendered by means of any known algorithm for Pólya curves, applied to the polygon $\mathbf{T}^{\alpha}$.

A great part of the total computational effort is devoted to the construction of the centrosymmetric matrix $C_{m,k}^{\alpha}$, which requires $O\left(\frac{m^3}{2}(k-2)\right)$ long operations. However, this construction needs to be performed only once, irrespective of the number of points to be rendered, n. Once the first point has been rendered, the additional computational effort required for rendering one more point on the curve is just that of performing one Pólya recursive algorithm, for each component.

Obviously, the total cost of this algorithm for GSP curves is bigger than the cost of any algorithm for Pólya curves, but the difference is negligible if $n \gg m$. In any case, it is much less than the cost of direct evaluation of the blending functions (3.3) and of expression (4.1).

Moreover, taking into account that

$$C_{m,k+1}^{\alpha} = C_{m,k}^{\alpha} + (I - A_m(\alpha))^k,$$

whenever, for a fixed value of α, not only the k-th GSP curve but also the $(k+1)$-th must be plotted, the procedure needs not be entirely repeated. Indeed, only a matrix product must be performed, with an additional cost of $m^3/2$ operations.

Finally, if the value of k can be chosen freely, allowing for $k = 2^p$, $p = 1, 2, \ldots$, will drastically reduce the computational cost, since

$$C_{m,2^p}^{\alpha} = C_{m,2^{p-1}}^{\alpha} + (I - A_m(\alpha))^{2^{p-1}} C_{m,2^{p-1}}^{\alpha}, \quad p = 1, 2, \ldots .$$

In this case, the construction of the matrix $C_{m,k}^{\alpha}$ is carried out in p steps with pm^3 long operations, where $p = \log_2 k$.

6. GRAPHICAL EXAMPLES

In the following examples the results that can be obtained by means of the careful joint adjustment of the two parameters are illustrated.

As we have pointed out before, Pólya curves generally allow us to reproduce the control polygon closely but they may exhibit undesired wiggles, due to instability phenomena. The GB curve scheme, on the other hand, does not suffer from such problems, yet it has some drawbacks too, that are related to its dependence on a discrete valued parameter, k. These facts were illustrated by examples in our previous paper [7].

Availability of the two handles at the same time, in the GSP scheme, actually permits us to overcome both limitations. For example, one can choose α sufficiently far from $-1/m$, so as to avoid wiggles, then increase k in order to go closer to the control polygon, thus obtaining a fully satisfactory result, as illustrated in the examples below.

In Fig. 1, the eighteen-point control polygon clearly suggests the shape of a bottle-neck, but the Bézier curve, illustrated in Fig. 1a, yields a definitely poor approximation. GB curve corresponding to $k=2$ (Fig. 1b) and Pólya curves corresponding to $\alpha = -0.03$ (Fig. 1c) and to $\alpha = -0.036$, provide better approximations, but still not quite satisfactory. Finally, combining the *optimal* values of the two parameters, an almost perfect smooth reproduction of the control polygon is obtained (Fig. 1f).

Fig. 2 illustrates how the wiggles that Pólya curves exhibit when α is close to $-1/m$ tend to be exaggerated by increasing of k (Fig. 2e and 2f) but a careful joint choice of α and k permits us to gain a perfect smooth reproduction even of a v-shaped control polygon with a sharp angle (Fig. 2d). This cannot be achieved by means of Pólya curves (Fig 2c) or of GB curves (Fig. 2b) alone.

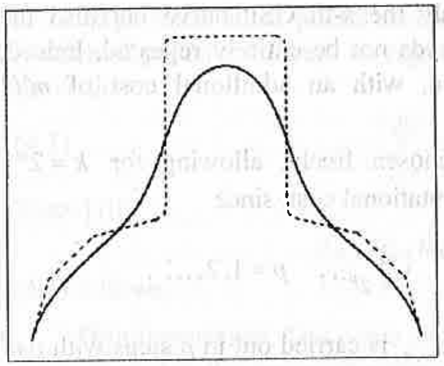

Fig. 1a – $\alpha = 0.$, $k = 1$, Actually yields a Bézier curve.

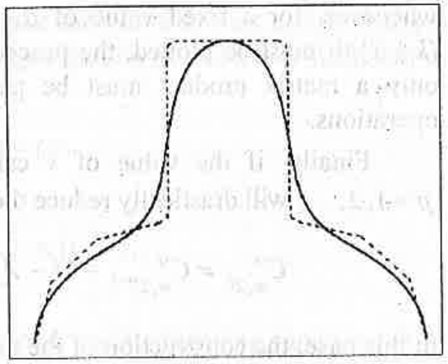

Fig. 1b – $\alpha = 0.$, $K = 2$, This is a GB curve.

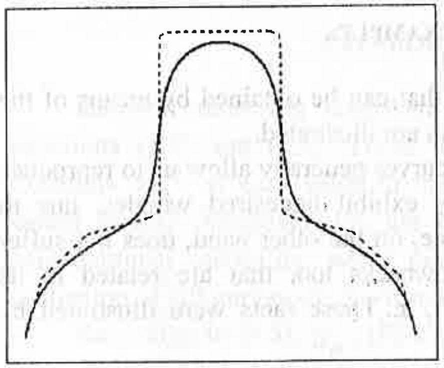

Fig. 1c – $\alpha = -0.03.$, $k = 1$, A Pólya curve.

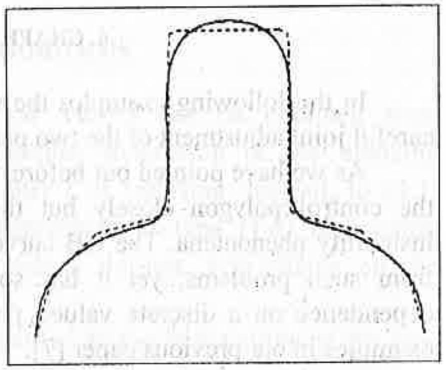

Fig. 1d – GSP curve for a $\alpha = -0.03$, $k = 2$.

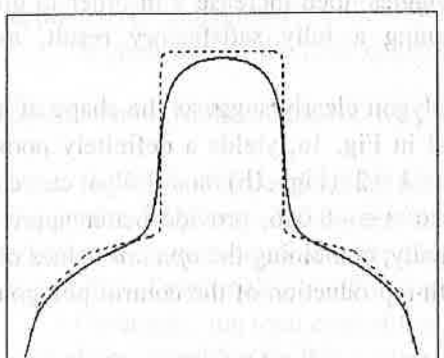

Fig. 1e – Pólya curve obtained for $\alpha = -0.036$, $k = 1$.

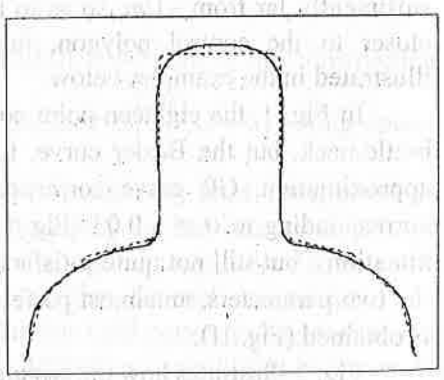

Fig. 1f – GSP curve obtained for a $\alpha = -0.036$, $k = 2$.

Fig. 1a – 1f. Different GSP-curves are obtained from the same 18-point bottle-neck-shaped control polygon by means of diffrent choices of the pair (α, k).

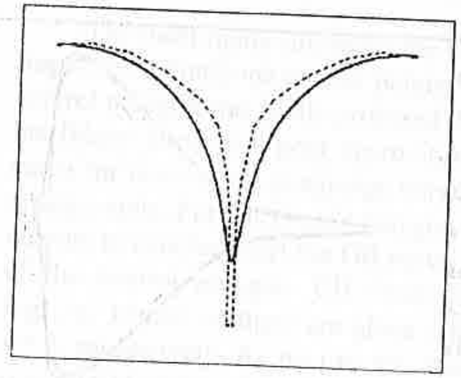

Fig. 2a – Bézier: $\alpha = 0.$, $k = 1$.

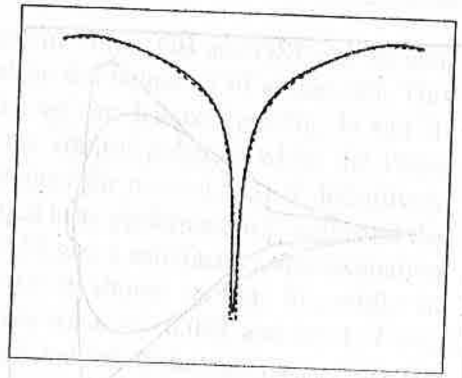

Fig. 2b – GB curve: $\alpha = 0.$, $k = 10$.

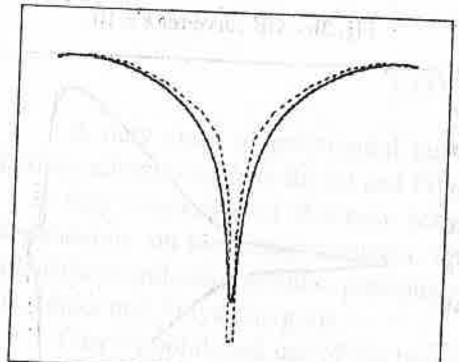

Fig. 2c – Pólya curve: $\alpha = -0.027.$, $k = 1$.

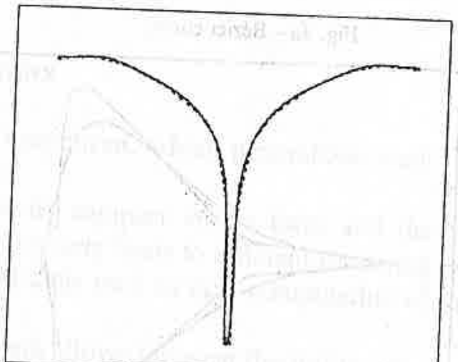

Fig. 2d – GSP curve: $\alpha = -0.027.$, $k = 10$.

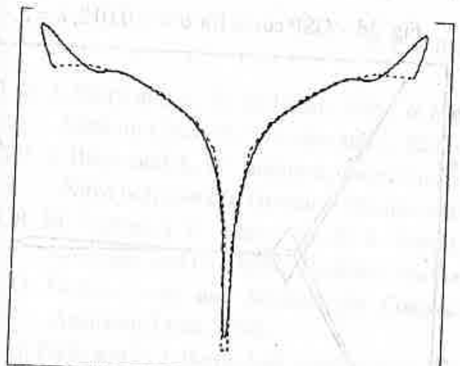

Fig. 2e – Pólya curve: $\alpha = -0.045.$, $k = 1$.

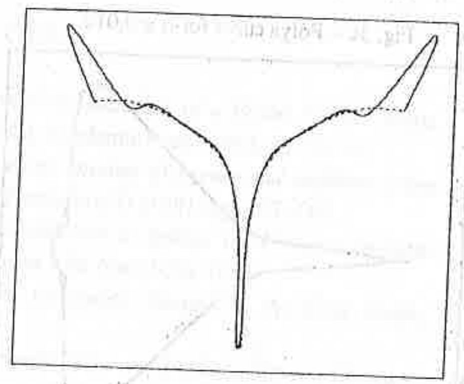

Fig. 2f – GSP curve: $\alpha = -0.045.$, $k = 2$.

Fig. 2a – 2f. Reproduction of a ν-shape. Adherence to the control polygon can be achieved by means of a careful choice of α and k.

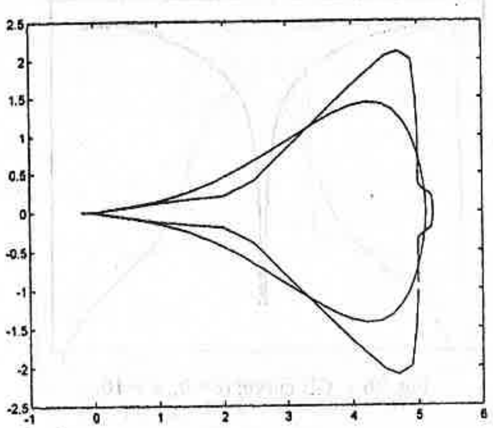

Fig. 3a – Bézier curve.

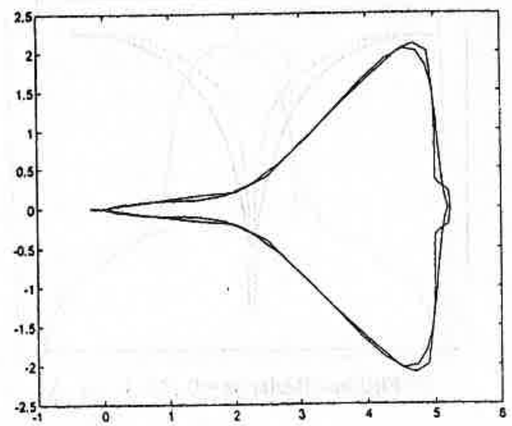

Fig. 3b – GB curve for $k = 10$.

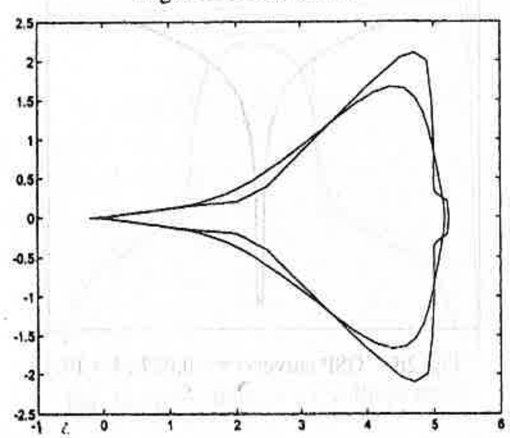

Fig. 3c – Pólya curve for $\alpha = 0{,}012$.

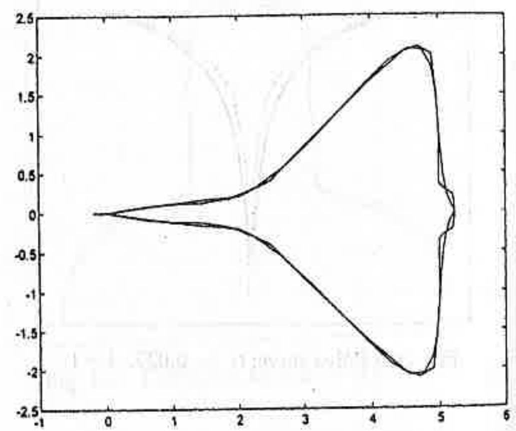

Fig. 3d – GSP curve for $\alpha = -0{,}012$, $k = 5$.

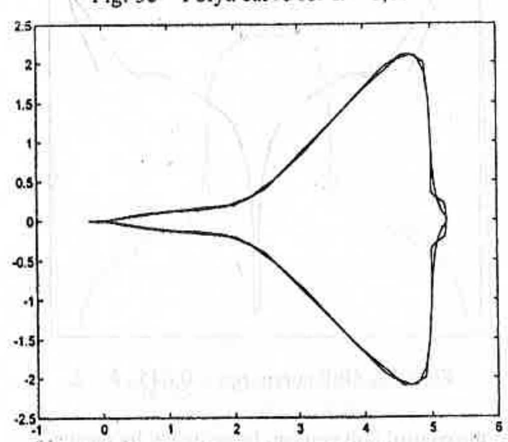

Fig. 3e – GSP curve for $\alpha = -0{,}012$, $k = 8$.

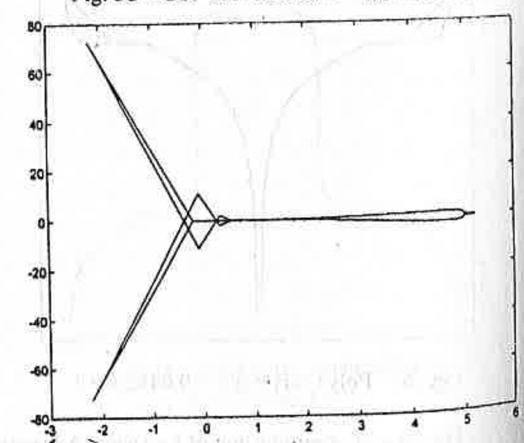

Fig. 3f – Pólya curve for $\alpha = -0{,}022$.

Figura 3a – 3f

The third figure illustrates the behaviour of Pólya, GB and GSP curves with respect to a fourtyone control points, located on the boundary of an aircraft. This control polygon was firstly proposed in [7]. As we can deduce from Fig. 3a and 3f, the Bézier curve is a poor approximant of the control polygon, while the Pólya curve for $\alpha = -0.022$ (Lagrange curve is obtained for $\alpha = -0.025$) is definitively unacceptable. For this control polygon graphical tests performed in [7] allowed the authors to conclude that the GB curve for $k = 10$ was a satisfactory approximation of the control polygon. GB curve for $k = 10$ is shown in Fig. 3b, while in Fig. 3c, 3d and 3e there are given GSP curves for $\alpha = -0.012$ and $k = 1$, $k = 5$, $k = 8$, respectively. As we can see, the optimal values of the two parameters are $\alpha = -0.012$ and $k = 8$. In fact, in this case the corresponding GSP curve shapes the control polygon better than the optimal GB curves for $k = 10$.

7. CONCLUSIONS

A new class of polynomial curves is introduced, which generalizes well-known schemes such as Bézier and Pólya ones.

Key properties of the new scheme are its compact vector form and the dependence on two free parameters. The first property leads to efficient rendering algorithms and also provides precious practical tools such as easy computation of its Bézier and Pólya polygons.

Careful combined use of the two parameters allows for great flexibility, thus permitting us to obtain a close reproduction of the control polygon, bypassing the unpleasant instability phenomena that Pólya curves sometimes exhibit.

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