REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome XXVII, N° 1, 1998, pp. 147–153

# INTERPOLATION BETWEEN FUNCTIONS OF MEANS

C. E. M. PEARCE, J. PEČARIĆ

## 1. INTRODUCTION

The familiar inequality between the geometric and arithmetic means of a pair of positive numbers has been translated by Seiffert [1] to a functional context in order to provide useful upper and lower bounds for certain integrals involving strictly monotone increasing functions. Seiffert gives the following result.

THEOREM A. *For* $0<a<b$, *let* $f:[a,b]\to R$ *be a Riemann-integrable, positive function and* $g:[(ab)^{1/2},(a+b)/2]\to R$ *a strictly monotonically increasing function. Then the inequality*

$$g((ab)^{1/2})<\int_a^b f(t)\,g(h(t))\,\mathrm{d}t \Big/ \int_a^b f(t)\,\mathrm{d}t<g((a+b)/2)$$

*holds, where* $h(t)=(t(a+b-t))^{1/2}$.

Arithmetic and geometric means arise as particular instances of a spectrum of means, the power mean of order $r$ of two positive numbers $a$, $b$ being defined by

$$M_r(a,b)=\left[\frac{1}{2}(a^r+b^r)\right]^{1/r},\quad r\neq 0,$$

$$M_0(a,b)=\sqrt{ab},\qquad r=0.$$

The provide the arithmetic mean through $A(a,b)=M_1(a,b)$ and the geometric mean through $G(a,b)=M_0(a,b)$.

This suggests that Seiffert's result may be generalized by making fuller use of the spectrum. This idea is implemented in Section 2, where we present a generalization of Theorem A.

In Section 3 we pursue a different development of this idea through the use of extended logarithmic means, which have found a useful unifying role in the

AMS Subject Classification: 41A05.

literature. The extended logaritmic mean of order $r$ of two positive numbers, $a$, $b$ is defined for $a \neq b$ by

$$L_r(a,b) = \left[\frac{b^r - a^r}{r(b-a)}\right]^{1/(r-1)}, \quad r \neq 0, 1,$$

$$L_0(a,b) = L(a,b) = \frac{b-a}{\log b - \log a},$$

$$L_1(a,b) = I(a,b) = \frac{1}{e}(b^b / a^a)^{1/(b-a)},$$

and for $a = b$ by

$$L_r(a,b) = b.$$

We note that $A(a,b) = L_2(a,b)$.

The bounds arising in Section 3 may be viewed as arising from the use of integral power means with a function $W(x) \equiv x$. For a positive, Riemann-integrable function $W:[a,b] \to R$, the integral power mean of order $r$ is defined by

$$M_r(W;a,b) = \begin{cases} \left[\dfrac{1}{b-a}\displaystyle\int_a^b \{W(x)\}^r \, dx\right]^{1/r}, & r \neq 0, \\ \exp\left[\dfrac{1}{b-a}\displaystyle\int_a^b \log W(x) \, dx\right], & r = 0. \end{cases}$$

In Section 4 we show that the results of Section 3 may be further extended to a class of positive, continuous functions $W$ for which either $W^r$ or $\log W$ has appropriate convexity or concavity properties.

## 2. POWER MEANS

In this section we establish the following generalization of Theorem A.

THEOREM 1. *For $0 < a < b$, let $f:[a,b] \to R$ be a positive, Riemann-integrable function. For*

$$A = \min\{M_r(a,b), A(a,b)\}, \quad B = \max\{M_r(a,b), A(a,b)\},$$

*let $g:[A,B] \to R$ be a strictly monotonic function and put $h(t) = M_r(t, a+b-t)$. If $r < 1$, then*

$$g(M_r(a,b)) < \int_a^b f(t)\, g(h(t))\, dt \Big/ \int_a^b f(t)\, dt < g(A(a,b)) \tag{2.1}$$



*when $g$ is increasing and the reverse inequalities hold when $g$ is decreasing. For $r > 1$ the inequalities are reversed.*

*Proof.* If $g$ is increasing, then

$$g(m) < \int_a^b f(t)\, g(h(t))\, dt \Big/ \int_a^b f(t)\, dt < g(M), \tag{2.2}$$

where $m = \min\limits_{t \in [a,b]} h(t)$ and $M = \max\limits_{t \in [a,b]} h(t)$.

Hence, we only need to prove in this case that $m = M_r(a,b)$ and $M = A(a,b)$ when $r < 1$ and that $M = A(a,b)$, $M = M_r(a,b)$ when $r > 1$. Again, since $M_r(a,b) = M_r(b,a)$, we can restrict our attention to the interval $[a, ((a+b)/2)]$.

Since $a + b - t > t$ and

$$h'(t) = \left[\frac{t^r + (a+b-t)^r}{2}\right]^{1/r-1} \cdot \frac{t^{r-1} - (a+b-t)^{r-1}}{2},$$

we have $h'(t) > 0$ for $r < 1$ and $h'(t) < 0$ for $r > 1$. Hence $m = h(a) = M_r(a,b)$ and $M = h((a+b)/2) = A(a,b)$ for $r < 1$, while for $r > 1$ we have $m = h((a+b)/2)$ and $M = h(a) = M_r(a,b)$. Thus (2.1) follows from (2.2).

A similar argument applies for $g$ decreasing.

## 3. EXTENDED LOGARITHMIC MEANS

THEOREM 2. *For $0 < a < b$, let $f:[a,b] \to R$ be a positive Riemann-integrable function. For*

$$A = \min\{L_r(a,b), A(a,b)\}, \quad B = \max\{L_r(a,b), A(a,b)\},$$

*let $g:[A,B] \to R$ be a strictly monotonic function and put $h(t) = L_r(t, a+b-t)$. If $r < 2$, then*

$$g(L_r(a,b)) < \int_a^b f(t)\, g(h(t))\, dt \Big/ \int_a^b f(t)\, dt < g(A(a,b))$$

*when $g$ is increasing and the reverse inequalities hold when $g$ is decreasing. For $r > 2$ the inequalities are reversed.*

*Proof.* As in the previous theorem, (2.2) holds for increasing $g$. Further, we have for $r \neq 0, 1$ that

$$(3.1)\quad \frac{h'(t)}{h(t)} = \frac{(2-r)[(a+b-t)^r - t^r] + rt(a+b-t)[(a+b-t)^{r-2} - t^{r-2}]}{(r-1)[(a+b-t)^r - t^r][a+b-2t]} =$$

$$= \frac{(2-r)(x^r - 1) + rx(x^{r-2} - 1)}{t(r-1)(x^r - 1)(x-1)},$$

where $x = (a+b-t)/t (>1 \text{ on } [a,(a+b)/2])$.

Let us consider the function

$$G(x) = (2-r)(x^r - 1) + rx(x^{r-2} - 1).$$

We have

$$G'(x) = (2-r)rx^{r-1} + r(r-1)x^{r-2} - r,$$

$$G''(x) = r(r-1)(2-r)x^{r-3}(x-1),$$

so that $G(1) = G'(1) = 0$.

Furthermore, $G''(x) > 0$ for $r \in (0,1) \cup (2,\infty)$, while $G''(x) < 0$ for $r \in (0,1) \cup (2,\infty)$. Therefore $G'(x) > 0$ for $r \in (-\infty,0) \cup (1,2)$ and $G'(x) < 0$ for $r \in (0,1) \cup (2,\infty)$. Hence $G(x) > 0$ for $r \in (-\infty,1) \cup (1,2)$ and $G(x) < 0$ for $x \in (0,1) \cup (2,\infty)$. Returning to (3.1), we can see that $h'(t) > 0$ for $r \in (-\infty,2) \setminus \{0,1\}$, that is, $h$ increases on $[a,(a+2)/2]$, while $h'(t) < 0$ for $r \in (2,\infty)$, that is, $h$ decreases on $[a,(a+b)/2]$.

Hence for $r \in (-\infty,2) \setminus \{0,1\}$ we have

$$m = h(a) = L_r(a,b), \quad M = h\left(\frac{a+b}{2}\right) = A(a,b)$$

and for $r \in (2,\infty)$

$$m = h\left(\frac{a+b}{2}\right) = A(a,b), \quad M = h(a) = L_r(a,b).$$

These results may be extended to the special cases $r = 0, 1$.

For $r = 0$ we have

$$h'(t) = \frac{2}{\log(a+b-t) - \log t} + \frac{(a+b-2t)(a+b)}{t(a+b-t)[\log(a+b-t) - \log t]^2} =$$

$$= \frac{2L(a+b-t,t)}{(a+b-2t)G^2(a+b-t,t)}\{A(a+b-t,t)L(a+b-t,t) - G^2(a+b-t,t)\} > 0.$$



That is, $h(t)$ is an increasing function on $[a,(a+b)/2]$. Thus

$$m = h(a) = L(a,b) \quad \text{and} \quad M = h\left(\frac{a+b}{2}\right) = A(a,b).$$

For $r = 1$, we have

$$h'(t) = \frac{2}{a+b-2t}\left\{\frac{a+b}{2} - \frac{a+b-2t}{\log(a+b-t) - \log t}\right\} > 0,$$

so we have again that $h$ is increasing on $[a,(a+b)/2]$ and so

$$m = h(a) = I(a,b), \quad M = h\left(\frac{a+b}{2}\right) = A(a,b).$$

The proof is completed as before.

## 4. INTEGRAL POWER MEANS

Finally, we introduce a function $W$ into our upper and lower bounds.

THEOREM 3. *Let $f:[a,b] \to R$ be a positive, Riemann-integrable function and $W:[a,b] \to R^+$ a positive, continuous function. For*

$$A = \min\left\{M_r(W;a,b), W\left(\frac{a+b}{2}\right)\right\}, \quad B = \max\left\{M_r(W;a,b), W\left(\frac{a+b}{2}\right)\right\},$$

*let $g:[A,B] \to R$ be a strictly monotonic function and put*

$$h(t) = M_r(W;t,a+b-t).$$

*If $g$ is an increasing function, then we have the inequality*

$$(4.1)\quad g(M_r(W;a,b)) < \int_a^b f(t)\, g(h(t))\, dt \Big/ \int_a^b f(t)\, dt < g\left(W\left(\frac{a+b}{2}\right)\right)$$

*if any of the following holds:*

i) $r > 0$ *and* $W^r$ *is concave;*

ii) $r < 0$ *and* $W^r$ *is convex;*

iii) $r = 0$ *and* $\log W$ *is concave.*

*Relation (4.1) applies with the inequalities reversed inequality if any of the following holds:*

iv) $r > 0$ *and* $W^r$ *is convex;*

v) $r < 0$ *and* $W^r$ *is concave;*

vi) $r = 0$ *and* $\log W$ *is convex.*

*If g is a decreasing function, these results hold with the inequalities reversed.*

*Proof.* Let g be increasing. For $r \neq 0$, we have

$$h'(t) = \frac{2}{r} \frac{1}{(a+b-2t)^2} \left[ \frac{1}{a+b-2t} \int_t^{a+b-t} [W(x)]^r \, dx \right]^{(1/r)-1} \times$$

$$\times \left\{ \frac{1}{a+b-2t} \int_t^{a+b-t} W^r \, dx - \frac{W^r(a+b-t) + W^r(t)}{2} \right\}.$$

We employ the wel-known Hadamard's inequality for convex function $\phi$, that is,

$$\frac{1}{v-u} \int_u^v \phi(x) \, dx < \frac{\phi(u) + \phi(v)}{2}, \tag{4.2}$$

while the reverse inequality is valid for a concave function $\phi$.

If either (i) or (ii) holds, then $h(t)$ is an increasing function on $[a, (a+b)/2]$ and

$$m = \min h(t) = h(a) = M_r(W; a, b),$$

while

$$M = \sup h(t) = h\left(\frac{a+b}{2}\right) = W\left(\frac{a+b}{2}\right).$$

Further, if either (iv) or (v) holds, we have that $h$ is a decreasing function on $[a, (a+b)/2]$ and that

$$m = \max h(t) = h(a) = M_r(W; a, b),$$

$$M = \inf h(t) = h\left(\frac{a+b}{2}\right) = W\left(\frac{a+b}{2}\right).$$

Using (2.2), we get (4.1).

For $r = 0$, we have

$$h'(t) = \frac{2}{a+b-2t} \exp\left[ \frac{1}{a+b-2t} \int_t^{a+b-t} \log W(x) \, dx \right] \times$$

$$\times \left\{ \frac{1}{a+b-2t} \int_t^{a+b-t} \log W(x) \, dx - \frac{\log W(a+b-t) + \log W(t)}{2} \right\},$$



that is, $h$ is increasing on $[a, (a+b)/2]$ if (iii) holds, while $h$ is decreasing on $[a, (a+b)/2]$ if (vi) holds, by Hadamard's inequality (4.2).

The proof is completed as before.

*Remark.* For $W(x) = x$, Theorem 3 gives Theorem 2.

## REFERENCE

1. H. J. Seiffert, *Werte Zwischen den geometrischen und den arithmetischen Mittel zweir Zahlen*, Elem. Math. **42** (1987), 105–107.

Received January 15, 1996.

*C. E. M. Pearce, Applied Mathematics Department*
*University of Adelaide, Australia*
*J. Pečarić Faculty of Textile Technology*
*University of Zagreb, Croatia*