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CVBEM FOR THE FLUID FLOW DETERMINED BY THE MOTION OF A DIRIGIBLE BALLOON IN A WIND STREAM

TITUS PETRILA

1. It is known that it is always possible to join a plane potential incompressible fluid flow to an analytic function, which is called the complex potential of the flow.

In fact, to determine this complex potential is synonymous with the integration of the system of P.D.E., with appropriate boundary conditions, which is governing the fluid flow. More precisely, this system of P.D.E. (also called the Euler system) can be written, in the simplest barotropic case, in the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0$$

$$\rho \frac{d\vec{v}}{dt} + \operatorname{grad} p = \vec{f}$$

$$p = g(\rho),$$

where the unknowns are ρ (the mass density), p (the pressure) and (u, v) (the velocity), while $\vec{f}$ is the known distribution of the external forces and g is a given function.

In the case of an incompressible and potential flow, the above system could be replaced by

$$\operatorname{div} \vec{v} = 0 \wedge \operatorname{rot} \vec{v} = 0$$

which, in 2-D problems, allows the introduction of a couple of real harmonic functions φ and ψ such that $\vec{v} = \operatorname{grad} \varphi = -\vec{k} \times \operatorname{grad} \psi$, while $f(x) = \varphi(x, y) + i\psi(x, y)$ is an analytic function representing the above-mentioned complex potential of the flow.

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For a flow past an obstacle (C), taking into account also the slip condition on the boundary of (C), i.e., $\vec{v}\cdot\vec{n}$ is known of boundary, the determination of the complex potential comes to an exterior Neumann or a Dirichlet problem for the Laplace operator (in terms of φ and ψ) with a certain behavior at infinity.

Let us consider the case when the obstacle (C) is a wing profile wich performs a general displacement (a "rototranslation") of parametres (1, m, ω) in our fluid.

Then the complex potential will be an analytic function in every point of the finite plane, having a singularity at infinity. More precisely, in the neighborhood of infinity the function $f(z)$ has a development of the form

$$f(z)=w_{\infty}z+\Gamma/2\pi\mathrm{i}\ln z+\ldots+a_0+a_1/z+\ldots,$$

where $w_{\infty}\equiv u_{\infty}-\mathrm{i}v_{\infty}$ is the complex velocity at infinity and Γ is a so-called circulation (the multiformity period of the real part φ of the complex potential f).

As regards the circulation Γ in the case of great importance in aerohydrodynamics, when the profile (C) has a sharp trailing edge z_F, where the "jump" of the semitangents is $\mu\pi, -1\le\mu<0$, this circulation must be chosen so that the boundedness of the velocity in the neighborhood of this point is ensured. More precisely, $\Gamma=Ll+Mm+N\omega$ (a so-called "Couchet rule"), where the coefficients L, M, N depend upon the given profile.

We remark now that, if instead of a complex potential f we would consider the complex velocity $w\equiv\dfrac{\mathrm{d}f}{\mathrm{d}z}$, then this will be a holomorphic function in the whole outside of the profile (C), which also includes the point of infinity. To be more exact, in the neighborhood of infinity, this function has the following development

$$w(z;t)=w_{\infty}+\frac{\Gamma}{2\pi\mathrm{i}}\frac{1}{z}+\frac{b_2}{z^2}+\frac{b_3}{x^3}+\ldots\,.$$

It is just the regularity of this complex velocity that suggests us the use of this function for CVBEM and not the complex potential f, as we would have been templet to. Of course, the boundary conditions must be written in terms of this complex velocity.

In what follows we want to determine the fluid flow induced by the general displacement (rototranslation) of parameters $(1,m,\omega)$ of our given profile (C), in the mass of the considered inviscid fluid. For more generality, we would suppose that the considered fluid has already an initial flow, named basic flow, of complex velocity $w_B(z)$, a flow which will be superposed on that produced by the displacement of the profile (C). We shall denote by $\zeta=\zeta(\beta),\beta\in[0,2\pi)$ the parametrical equation of the Jordan rectifiable curve C, which is a 2π periodical function, bounded and derivable in $[0,2\pi]\setminus\{\beta_0\}$, where $\dot{\zeta}(\beta)\neq 0$ and $\zeta(\beta)<M$, M being a finite constant. The angular point is given by $z_F=\zeta(\beta_0)$.

Using then our previous results, we can state that the functions $w_B(z)$ belong to a class (a) having the following properties [1]:

1a. They are holomorphic in the plane D_1, except for a finite number of points z_q, which are singular points for these functions. Let us denote by D_1^* the domain D_1 from which one has taken off these singular points.

2a. The circulaton Γ_B must be equal to the sum of the circulations due to the presence of the given singularities.

As regards the unknown function $w(z)$, the complex velocity of the resultant flow obtained by the above-mentioned superposition, it belongs to a class (b) having the following properties:

1b. They are holomorphic functions in $D=D_1/(C)$, except for the same points z_q, which are for them singular points of the same nature as for $w_B(z)$; at infinity their behavior is identical with that of $w_B(z)$, i.e.,

$$\lim_{|z|\to\infty}w(z)=w(\infty)=w_B(\infty).$$

2b. In the neighborhood of the trailing edge $z_F=\zeta(\beta_0)\in C$, where the semitangents angle is $\pi-\mu\pi$, we have

$$w(z)=(z-z_F)^{\mu/(1-\mu)}g(z),g(z_p)\neq 0.$$

3b. In the points of the curve C, the functions $w^*(\zeta(\beta))$ belong to the class H^*, i.e., they are Holderian functions on C except the angular point $z_F=\zeta(\beta_0)$, in whose neighborhood one has

$$w(\zeta(\beta))=\frac{w^*(\zeta(\beta))}{[\zeta(\beta)-\zeta(\beta_0)]^{\mu/(1-\mu)}},$$

where $w^*\in H_0$ in the same neighborhood, which means that $w^*(\zeta(\beta))$ is separately Holderian on the upper side and on the lower side of the profile in the neighborhood of $z_F=\zeta(\beta_0)$.

4b. In the points of the curve C they satisfy, except at the angular point, the following boundary condition:

There is a real continuous function $v(\beta)$ such that for every $\beta\in[0,2\pi]\setminus\{\beta_0\}$ one has

$$w(\zeta(\beta))=\upsilon(\beta)\frac{\zeta(\beta)}{|\zeta(\beta)|}+l+\mathrm{i}m+\mathrm{i}\omega[\zeta(\beta)-z_A],$$

where $z_A \in (C)$ and $l(t), m(t), (t)$ are the given functions of time determining the rototranslation of the profile (C).

5b. They fulfill the equality $\int_C w(z)\,dz = \Gamma$, where the circulation of the flow Γ is chosen so that one has the boundedness of the velocity in z_F, i.e., $\Gamma = Ll + Mm + Nn$, where the coefficients *L, M, N* are given with the obstacle (C).

Now let us consider the difference $w(z) - w_B(z)$. This function, known together with $w(z)$, is a holomorphic function in the whole outside of (C). Using the Cauchy formula for this function, we get

$$w(\zeta) = w_B(\zeta) - \frac{1}{2\pi i}\int_C \frac{w(z)}{z-\zeta}dz + \frac{1}{2\pi i}\int_C \frac{w_B(z)}{z-\zeta}dz \text{ for } \zeta \in D.$$

This integral equation, which is a singular one with Cauchy kernel, will be the main instrument for the development of the CVBEM. More precisely, this equation will be considered the integral representation joined to the boundary problem formulated in terms of the holomorphic function $w(z) - w_B(z)$.

Finally, in order to use the boundary conditions on C, we perform

$$\zeta \to \zeta^* = \zeta(\beta^*) \in C \setminus \{z_F\}$$

and so we get

$$w(\zeta(\beta^*)) = w_B(\zeta(\beta^*)) - \frac{1}{\pi i}\int_C^{2\pi} \frac{w(\zeta(\beta)) \cdot \zeta(\beta)}{\zeta(\beta) - \zeta(\beta^*)}d\beta + \frac{1}{\pi i}\int_C^{2\pi} \frac{w_B(\zeta(\beta)) \cdot \zeta(\beta)}{\zeta(\beta) - \zeta(\beta^*)}d\beta.$$

This is the boundary integral equation which could be used in a BEM. But our CVBEM avoids this equation which seems to be a considerable simplification.

By separating the real parts of both sides of the singular integral equation, we get a Fredholm integral equation, which, under our assumptions, has a continuous kernel. Studying the existence of the solutions of this equation by Fredholm alternative we can state that this equation has a unique solution which fulfills the circulation condition.

Considering then a system of nodal points $z_0, z_1, \ldots z_{F-1}, z_F z_{F+1}, \ldots, z_n \equiv z_0$ on the curve C together with the system of the piecewise interpolating Lagrange functions on each arc C_j (linking the points z_{j-1} and z_j), a system which takes into account the behavior in the neighborhood of z_F, we can write

$$\widetilde{w}(\zeta(\beta)) - w_B(\zeta(\beta)) = \sum_{j=1}^{n} (w_j - w_{Bj})\, L_j,$$

where

$$L_j(\zeta(\beta)) \quad \text{for} \quad j \neq F-1, F, F+1$$

have the expressions

$$L_j(t) = \begin{cases} (t - z_{j-1})/(z_j - z_{j-1}) & t \in C_j \\ (t - z_{j+1})/(z_j - z_{j+1}) & t \in C_{j+1} \\ 0 & \text{otherwise,} \end{cases}$$

while for $j = F-1, F, F+1$ they are

$$L_{F-1}(t) = \begin{cases} (t - z_{F-2})/(z_{F-1} - z_{F-2}), & \text{for } t \in C_{F-1} \\ ((t - z_f)/(z_{f-1} - z_f))^{1/(1-\mu)}, & \text{for } t \in C_F \\ 0, & \text{otherwise,} \end{cases}$$

$$L_F(t) = \begin{cases} 1 - ((t - z_F)/(z_{F-1} - z_F))^{1/(1-\mu)}, & \text{for } t \in C_F \\ 1 - ((t - z_F)/(z_{F+1} - z_F))^{1/(1-\mu)}, & \text{for } t \in C_{F+1} \\ 0, & \text{otherwise,} \end{cases}$$

$$L_{F+1}(t) = \begin{cases} (t - z_{F+2})/(z_{F+1} - z_{F+2}), & \text{for } t \in C_{F+2} \\ ((t - z_f)/(z_{F+1} - z_F))^{1/(1-\mu)}, & \text{for } t \in C_{F+1} \\ 0, & \text{otherwise.} \end{cases}$$

Using subsequently the general calculations already performed for $\widetilde{L}_j(z)$, i.e., for $L_{jk} = \widetilde{L}_j(z_k)$ [2], if $w(z_k) - w_B(z_k) \equiv u_k - iv_k$ and $L_{jk} \equiv M_{kj} + iN_{kj}$, we are led again to the real algebraic homogeneous system

$$u_k = \sum_{j=1}^{n} M_{kj} u_j + \sum_{j=1}^{n} N_{kj} v_j$$

$$v_k = \sum_{j=1}^{n} M_{kj} v_j + \sum_{j=1}^{n} N_{jk} u_j,$$

which will be completed, in this case, by the complex equation

$$\sum_{j=1}^{n} w_j \int_C Lj(\zeta)\,d\zeta = \Gamma \text{ or, equivalently, by}$$

$$\sum_{j=1}^{n} u_j \operatorname{Re}\int_C Lj(\zeta)\,d\zeta + v_j \operatorname{Im}\int_C Lj(\zeta)\,d\zeta = \Gamma$$

$$\sum_{j=1}^{n} u_j \operatorname{Im}\int_C Lj(\zeta)\,d\zeta = \sum_{j=1}^{n} v_j \operatorname{Re}\int_C Lj(\zeta)\,d\zeta.$$

These last two real equations allow us to determine an unique solution of the above homogeneous system which also includes the data on C. This unique solution, once introduced in the integral representation of the problem (i.e., in our case the Cauchy formula), leads to the complete determination of the complex velocity in every point of the flow domain.

3. Let us now consider the fluid flow produced by the motion of a self-propelled dirigible ballon in a uniform stream of wind whose velocity is *a priori* given.

We assume that both the dirigible motion and the velocity of the wind stream depend explicitly on time. Besides the unsteady behavior, the mentioned flow of the inviscid and incompressible air is supposed to be plane and potential. Neither external forces nor the influence of the ground are considered, the dirigible being all the time at a sufficiently great distance from it.

As concerns the contour of the dirigible balloon, it could be expressed by an explicit equation of the type

$$x + \mathrm{i}y = \frac{8}{3\left[1-\left(1-\dfrac{2}{s+1\pm \mathrm{i}\sqrt{2-s-s^2}}\right)^k\right]} - k, \qquad s \in [-2,1].$$

This equation implies, besides the symmetry of the configuration *vs.* the real axis (i.e., the axis of dirigible), the existence of a trailing edge, located at the point of the abscise $x = k$ and where the angles of the semitangents with the real axis are respectively $\pm k\pi$.

In fact, the above profile is of Karmann-Trefftz type [4], the connection between the parameter μ of the preceding section and the just introduced parameter k being given by $\dfrac{\mu}{1-\mu} = \dfrac{1-k}{k}$. In the sequel we shall use the value $\dfrac{4}{3}$ for $k\left(k = \dfrac{4}{3}\right)$. As regards the stream of wind (the basic flow in terms of the previous section), it will be defined by its complex velocity $W_B(z;t) \equiv u_B - \mathrm{i}v_B = (2t+1) - \mathrm{i}\,3t^2$, while the displacement of the dirigible is defined by $l(t) = -3t^2$, $m(t) = t$, where $t = 0, 1, \ldots$ (the successive time instants) (see Fig. 1).

As to the value of the circulation Γ, which has to ensure the boundedness of the velocity in the neigborhood of the sharp trailing edge, it will be established by considering, instead of the flow produced by the dirigible motion, with the velocity $(l(t), m(t))$ in a fluid initialy at rest, the "dual problem", i.e., that of an opposite fluid stream of velocity $(-l(t), -m(t))$ past our profile (dirigible) supposed now

fixed. By cumulating this "attack" velocity $(-l(t), -m(t))$ with the velocity of the wind (basic) stream (u_B, v_B), the Jukovski hypothesis leads to the following value of the circulation $\Gamma = -6\pi(v_B - m)$.

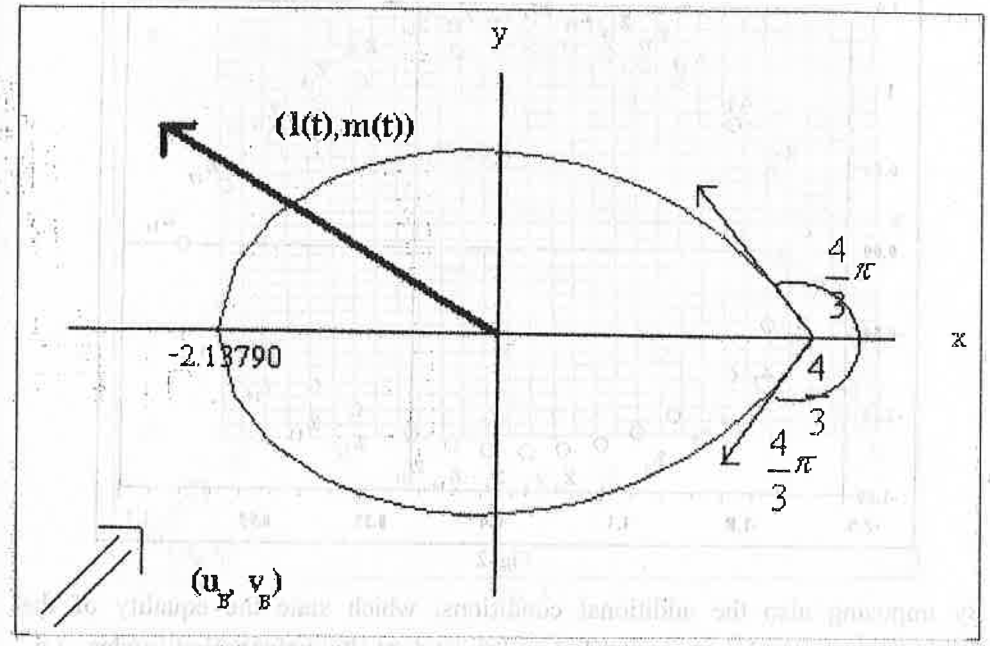


Fig. 1

We have taken into consideration above the fact that the image of our profile (dirigible), through the mapping $\dfrac{z-\frac{4}{3}}{z+\frac{4}{3}} = \left(\dfrac{Z-1}{Z+1}\right)^{\frac{4}{3}}$, is a circumference centered at $\left(-\dfrac{1}{2}, 0\right)$ of radius $\dfrac{3}{2}$ and whose point $Z = 1$ corresponds to the sharp trailing edge.

The kinematic (slip) condition at the points of the dirigible contour will be written as $\dfrac{v + v_B - m}{u + u_B - l} = \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_C$, where $w = u - \mathrm{i}v$ is the looked-for complex (relative) velocity of the fluid flow *vs.* the system of axis xoy, rigidly linked to the profile (dirigible).

As regards the nodes $z_1, z_2, z_3, \ldots, z_{30}$ $(z_{31} = z_1)$ chosen counterclockwise on the contour of the dirigible, they are obtained by allowing the real parameter of the explicit contour equation to take the values –2, –1.9, –1.7, –1.4, –1.2, –1.0, –0.8, –0.6, –0.4, –0.2, 0.0, –0.5, –0.7, 0.9, 1; the leading edge is the node $z_1 = -2.13790$, while the trailing edge is the node $z_F, z_{16} = 1.3333$ (see Fig. 2).

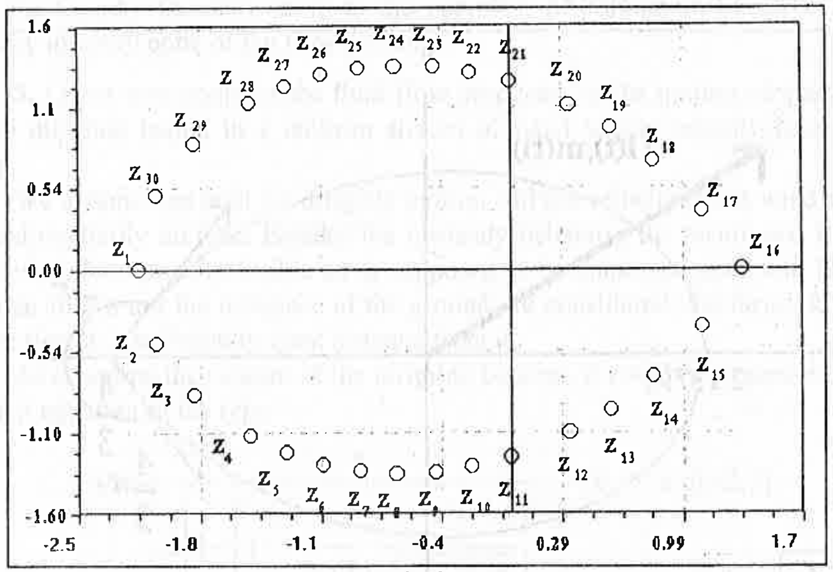


Fig. 2

By imposing also the additional conditions, which state the equality of the flow velocity at the sharp trailing edge and at its neighboring nodes, i.e., $u_{F-1} = u_F = u_{F+1}$ and $v_{F-1} = v_F = v_{F+1}$ with $F = 16$ (a compulsory requirement for avoiding some logarithmic singularities in calculation of $\widetilde{L}_F(z_{F-1})$ and $\widetilde{L}_F(z_{F+1})$), we are led to the solving of a linear algebraic system of 60 + 2 (circulation condition) real equations with 56 unknowns.

Since the slip condition written at all the 27 remaining nodes ($j \neq F-1$, F, $F+1$) and at z_{F-1} (or z_{F+1}) that means $\dfrac{v_j - v_B - m}{u_j + u_B - l} = \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{z=z_j}$, $j = \overline{1,30}$

$j \neq 16{,}17$, eliminate v_j in the favor of u_j, we are led again to an overdetermined nonhomogeneous system but this time of 62 equation with 29 unknowns.

The Gauss elimination method (using Borland C++ Compiler) allows us to find the unique solution of this system. Once it is obtained the value $u_j - \mathrm{i}v_j = \widetilde{W}(z_j)$ at the node $z_j \in C, j = \overline{1,30}$, we can proceed to the determining of the unknown function $W(z) \approx W^*(z) = \sum_{j=1}^{3} W_j \widetilde{L}_j(z)$. This will be done at the mesh points of a squared neighborhood, of size $[-5,5] \times [-5,5]$, of the profile

(dirigible), both the x – and y – steps of the respective mesh being equal to $\frac{1}{3}$, which means 961 points (see Fig. 3).

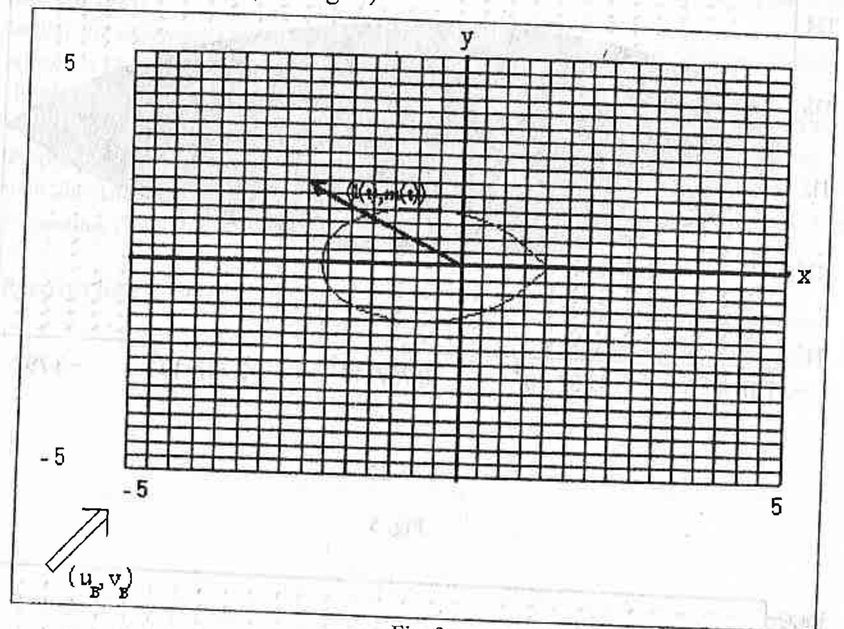
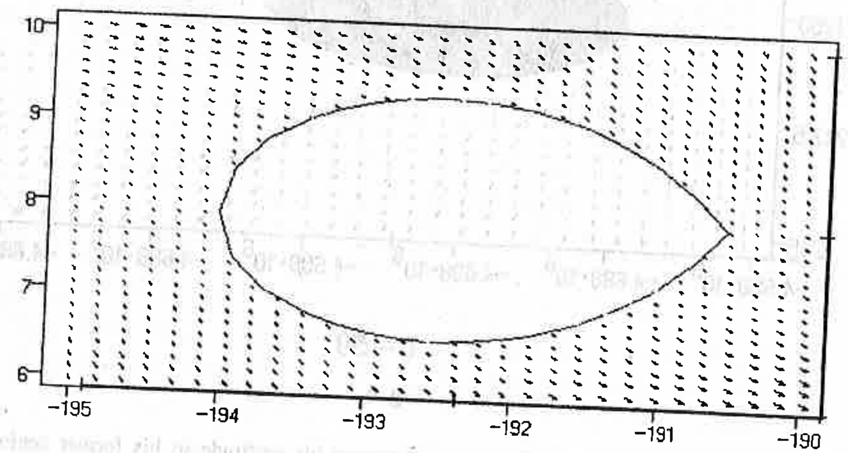


Fig. 3

Finally, the (absolute) velocity of the resultant fluid flow *vs.* a fixed system of axes, obtained by the above-mentioned superposition, will be determined by calculating the vector $\vec{V}(u+l, v+m)$ in the same mesh points and at different time moments (see Fig. 4, 5 and 6).



t = 4

Fig. 4

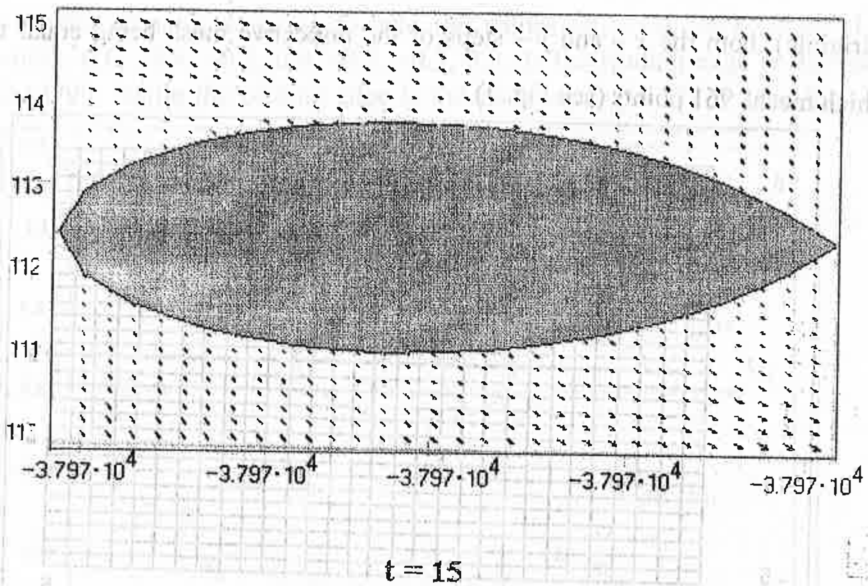


Fig. 5

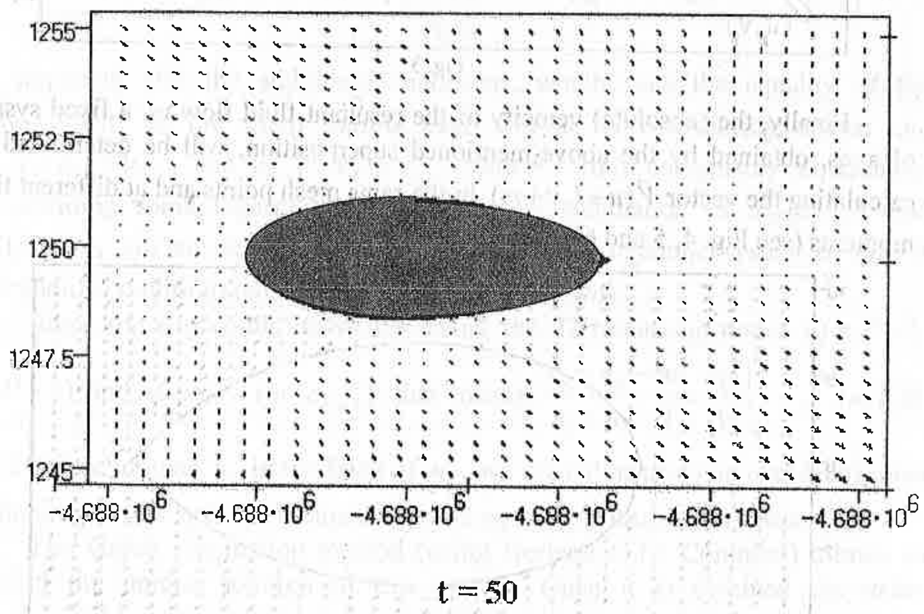


Fig. 6

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Department of Mathematics
"Babeş-Bolyai" University
3400 Cluj-Napoca
Romania