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ON APPROXIMATION BY BINOMIAL OPERATORS OF TIBERIU POPOVICIU TYPE

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1. INTRODUCTION

1. It is known that an important class of polynomial sequences $(p_m)_{m\geq 0}$ occurring in combinatorics and analysis is represented by the sequences of *binomial type* (B.T.), for which we have $p_0 = 1$, $\deg p_m = m$ and the following equalities

$$p_m(u+v) = \sum_{k=0}^{m} \binom{m}{k} p_k(u)\, p_{m-k}(v) \tag{1.1}$$

are identically satisfied in u and v for any nonnegative integer m.

One can obtain sequences of B.T. by using *generating functions methods* and by using *operatorial* or *umbral methods*.

It is known (cf., e.g., A. M. Garsia [2]) that (p_m) is a sequence of B.T. if and only if it is defined by a generating relation

$$[\phi(t)]^x = e^{x\varphi(t)} = \sum_{m=0}^{\infty} p_m(x)\frac{t^m}{m!}, \tag{1.2}$$

where

$$\phi(t) = 1 + a_1t + a_2t^2 + \ldots,\quad \varphi(t) = c_1t + c_2t^2 + \ldots\quad (c_1 \neq 0). \tag{1.3}$$

Sequences of B.T. have remarkable algebraic-combinatorial properties and many practical applications.

In 1931 the great Romanian mathematician Tiberiu Popoviciu [12] already had the wonderful idea to indicate a method for constructing linear polynomial operators, useful in constructive theory of functions, by means of sequences of B.T.

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If in the identity (1.1) we set $u = x$ and $v = 1 - x$, we obtain the identity

$$\sum_{k=0}^{m} \binom{m}{k} p_k(x)\, p_{m-k}(1-x) = p_m(1). \tag{1.4}$$

It suggests the introduction of an operator T_m, of Tiberiu Popoviciu [12], associated to a function $f : [0,1] \to R$, by means of the formula

$$(T_m f)(x) = \frac{1}{p_m(1)} \sum_{k=0}^{m} \binom{m}{k} p_k(x)\, p_{m-k}(1-x)\, f\left(\frac{k}{m}\right), \tag{1.5}$$

where $x \in [0,1]$ and $m = \mathbb{N}$, with the assumption: $p_m(1) \neq 0$.

It should be mentioned that, in fact, in front of the above sum he chose the factor $1/a_m$, but it is easy to see that we have $a_m = p_m(1)$ (we only have to replace in (1.2) $x = 1$ and then to identify the coefficients of t^m).

According to a result of Tiberiu Popoviciu [12], found later also by P. Sablonnière [16], we have $p_n(x) \geq 0$ on $[0,1]$ $(n = 0, 1, \ldots, m)$ if and only if the coefficients c_k from (1.3) are nonnegative. In this case the linear operator T_m is of positive type.

It is obvious that in the case of binomial polynomials $p_r(x) = e_r(x) = x^r$ $(r \in \mathbb{N}_0)$ T_m reduces to the operator B_m of Bernstein.

In this paper we shall consider some more general binomial-type operators and we shall study their approximation properties, including the estimation of the orders of approximation by means of the first and second moduli of smoothness, as well as representations of the remainder term in approximation formulas by the Tiberiu Popoviciu-type operators.

2. USE OF UMBRAL METHODS FOR OBTAINING BINOMIAL-TYPE POLYNOMIALS

2. The umbral methods are associated with the names of S. Pincherle, E. T. Bell, J. L. W. Jensen, J. I. Sylvester, J. F. Steffensen, G.-C. Rota, R. Mullin, S. Roman, A. M. Garsia, and others.

The operators Ω, considered in 1902 by Jensen [6], include the *shift operator* E^a, defined by $(E^a f)(x) = f(x+a)$, the *central mean operator* μ_h, defined by $(\mu_h f)(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$ and the *integration operator*.

An operator T which commutes with all shift operators is called a *shift-invariant operator*, i.e., $TE^a = E^a T$.

A special class of omega operators is represented by the *theta operators* θ (a term introduced in 1927 by Steffensen [23]) or *delta operators*, denoted by Q (a term suggested in 1956 by E. B. Hildebrand [5]) and used intensively by G.-C. Rota and his collaborators ([2], [11], [14] and [15]).

A theta operator θ is a shift-invariant operator for which θe_1 is a nonzero constant.

Such operators possess many of the properties of the derivative operator D. They are sometimes called *reductive operators*.

Here are some typical examples: the forward, backward and central difference operators $\Delta_h, \nabla_h, \delta_h$, the prederivative operator $D_h = \Delta_h / h$ (we consider that D_0 is the derivative operator D). Another example is represented by the Abel operator $A_a = DE^a = E^a D$, which in the case $p_n(x;a) = x(x-na)^{n-1}$ leads us to the formula: $A_a p_n(x;a) = nx(x-(n-1)a)^{n-2}$.

It should be noted that each theta operator θ can be expressed as a power series in the derivative operator (see [24] and [2]):

$$\theta = \sum_{k=1}^{\infty} c_k D^k \quad (c_1 \neq 0).$$

It is easy to see that: (i) for every theta operator we have $\theta c = 0$, where c is a constant; (ii) if p_n is a polynomial of degree n, then $\theta\, p_n$ is of degree $n-1$.

3. A great part of applied mathematics is concerned with the study of various special polynomials.

A sequence of polynomials (p_m) is called by I. M. Sheffer [17] and by Gian-Carlo Rota [15] and his collaborators, *the sequence of basic polynomials for a theta operator* θ if we have: (i) $p_0(x) = 1$; (ii) $p_n(0) = 0$ $(n \geq 1)$; (iii) $\theta p_n = n p_{n-1}$.

These polynomials were called by Steffensen [24] *poweroids*, considering that they represent an extension of the mathematical notion of power.

The following two results can be easily proved (see [11]): (i) if (p_n) is a basic sequence of polynomials for some theta operator, then it is a B.T. sequence; (ii) if (p_n) is a sequence of B.T., then it is a basic sequence for a theta operator.

It is easy to show, by induction, that every theta operator has a unique sequence of basic polynomials associated with it.

Steffensen [24] was the first who observed that the property of the polynomial sequence $e_n(x) = x^n$ to be of binomial type can be extended to an arbitrary sequence of basic polynomials associated to a theta operator.

Here are some illustrative examples: (i) if θ is the derivative operator D, then $p_m(x) = x^m$; (ii) if θ is the prederivative operator $D_h = h^{-1}\Delta_h$, then we arrive at the factorial power: $p_m(x)=x(x-h)\dots(x-(m-1)h)$; (iii) if $\theta=E^{\beta}\dfrac{1-E^{-\alpha}}{\alpha}$, then we have $p_m(x;\beta,\alpha)=x(x+\alpha+\beta m)^{[m-1,-\alpha]}$.

4. The classical Taylor formula can be extended to arbitrary theta operators.

If T is a shift-invariant operator and (p_n) is the corresponding basic polynomial sequence, then we have

$$T=\sum_{k=0}^{\infty}\frac{a_k}{k!}\theta^k,\ \text{where}\ a_k=[Tp_k(x)]_{x=0}.$$

The Taylor expansion formula for a polynomial P_m, of degree m, is given by

$$P_m(x+y)=\sum_{k=0}^{\infty}\frac{(\theta^k P_m)(y)}{k!}p_k(x), \tag{2.1}$$

where θ is a theta operator with the basic sequence (p_k).

Examples. If $\theta=A_{-\beta}=E^{-\beta}D$, then we obtain the Abel basic polynomials $p_k(x)=x(x+k\beta)^{k-1}$ and the Taylor expansion becomes

$$P_m(x+y)=\sum_{k=0}^{m}\frac{P_m^{(k)}(y-\beta k)}{k!}x(x+k\beta)^{k-1}. \tag{2.2}$$

If we choose $P_m(x)=x^m$, we obtain

$$(x+y)^m=\sum_{k=0}^{m}\binom{m}{k}x(x+k\beta)^{k-1}(y-k\beta)^{m-k}, \tag{2.3}$$

which is the Abel classical combinatorial identity.

When we select $P_m(x)=x(x+m\beta)^{m-1}$ we have $(\theta^k P_m)(y)=$ $=m^{[k]}(y+(m-k)\beta)^{m-1}$ and formula (2.2) leads us to the Abel-Jensen combinatorial formula

$$(x+y)(x+y+m\beta)^{m-1}=\sum_{k=0}^{m}\binom{m}{k}x(x+k\beta)^{k-1}y(y+(m-k)\beta)^{m-1-k}, \tag{2.4}$$

which can be obtained from (1.1) if we use the following sequence of binomial type: $p_n(t)=t(t+n\beta)^{n-1}$.

If we consider a generalized Vandermonde formula of the following form

$$(x+y+m\beta)^{[m,h]}=\sum_{k=0}^{m}\binom{m}{k}x(x+k\beta-h)^{[k-1,h]}(y+(m-k)\beta)^{[m-k,h]},$$

and we take $h=0$, we obtain the Abel-Jensen combinatorial formula

$$(x+y+m\beta)^m=\sum_{k=0}^{m}\binom{m}{k}x(x+k\beta)^{k-1}(y+(m-k)\beta)^{m-k}. \tag{2.5}$$

We mention that Cheney and Sharma have used in [1] the combinatorial formulas (2.4) and (2.5) for constructing two Bernstein-type operators P_m and Q_m, depending on the nonnegative parameter $\beta=o(n^{-1})$.

3. GENERALIZED BINOMIAL OPERATORS OF TIBERIU POPOVICIU TYPE

5. Let (p_n) be a basic sequence of polynomials for some theta operator. It follows that it is a B.T. sequence and we can write the identity (1.4).

Assuming that $p_m(1)\neq 0$, for any $m\in\mathbb{N}$, we define an operator of binomial type $T_m^{\gamma,\delta}$, associated to a function $f:[0,1]\to\mathbb{R}$, by the following formula

$$T_m^{\gamma,\delta}(f(t);x=(T_m^{\gamma,\delta}f)(x)=$$
$$=\frac{1}{p_m(1)}\sum_{k=0}^{m}\binom{m}{k}p_k(x)\,p_{m-k}(1-x)\,f\left(\frac{k+\gamma}{m+\delta}\right), \tag{3.1}$$

where $x\in[0,1]$ and γ and δ are parameters satisfying the relations: $0\le\gamma\le\delta$.

As we have pointed out, we have $p_j(x)\ge 0$ on $[0,1]$, for $j=0,1,\dots,m$, if the coefficients c_k from (1.3) are nonnegative.

It is obvious that our operator reproduces the constants, since according to (1.4) we have $T_m^{\gamma,\delta}e_0=e_0$.

Now we want to see if we can choose the parameters γ and δ such that the polynomial $T_m^{\gamma,\delta}f$ is interpolatory at the endpoints of the interval [0, 1].

It is easily verified that we have

$$(T_m^{\gamma,\delta}f)(0)=f\left(\frac{\gamma}{m+\delta}\right),\quad (T_m^{\gamma,\delta}f)(1)=f\left(\frac{m+\gamma}{m+\delta}\right).$$

Consequently, if we take $\gamma=\delta=0$ we can see that the Tiberiu Popoviciu polynomial $T_mf=T_m^{0,0}f$ is interpolatory in $x=0$ and $x=1$, that is

$$(T_mf)(0)=f(0),\quad (T_mf)(1)=f(1). \tag{3.2}$$

In order to study the convergence of the sequence $(T_m^{\gamma,\delta}f)$, we also need to find its value for the monomials e_1 and e_2.

In the case $f=e_1$ we have

$$(T_m^{\gamma,\delta}e_1)(x)=\frac{1}{p_m(1)}\sum_{k=0}^{m}\frac{k+\gamma}{m+\delta}\binom{m}{k}p_k(x)\,p_{m-k}(1-x)=$$

$$=\frac{m}{m+\delta}(T_me_1)(x)+\frac{\gamma}{m+\delta}(T_me_0)(x)=\frac{mx+\gamma}{m+\delta},$$

since in a recent paper [22] it has been proved that $T_me_1=e_1$. Hence we have

$$T_m^{\gamma,\delta}e_0=e_0,\quad (T_m^{\gamma,\delta}e_1)(x)=x+\frac{\gamma-\delta x}{m+\delta}. \tag{3.3}$$

Going on to $e_2(x)=x^2$ we obtain

$$(T_m^{\gamma,\delta}e_2)(x)=\frac{1}{p_m(1)}\sum_{k=0}^{m}\left(\frac{k+\gamma}{m+\delta}\right)^2p_k(x)\,p_{m-k}(1-x)=$$

$$=\frac{1}{(m+\delta)^2}[m^2(T_me_2)(x)+2m\gamma(T_me_2)(x)+\gamma^2].$$

Therefore

$$(T_m^{\gamma,\delta}e_2)(x)=\frac{\gamma^2}{(m+\delta)^2}+\frac{2m\gamma x}{(m+\delta)^2}+\frac{m^2}{(m+\delta)^2}(T_me_2)(x). \tag{3.4}$$

It remains to evaluate $(T_me_2)(x)$.

In this sense we mention two results.

The first one was found by the former graduate student of the first author – C. Manole [9]:

$$(T_me_2)(x)=x^2+\frac{x(1-x)}{m}+x(1-x)\,a_m, \tag{3.5}$$

where

$$a_m=\frac{m-1}{m}[1-p_m^{-1}(\theta')\,p_{m-2}(1)], \tag{3.6}$$

θ' being the Pincherle derivative (see, e.g. [11]) of the theta operator θ, for which (p_m) is a basic sequence.

The second result has been recently obtained by P. Sablonnière [16]

$$(T_me_2)(x)=x^2+\frac{x(1-x)}{m}b_m,\quad b_m=1+(m-1)\frac{r_{m-2}(1)}{p_m(1)}, \tag{3.7}$$

the sequence (r_m) being generated by the expansion

$$h''(t)\,e^{xh(t)}=\sum_{m=0}^{\infty}r_m(x)\frac{t^m}{m!}. \tag{3.8}$$

If we take into account (3.3), (3.4) and (3.5)–(3.6) or (3.7)–(3.8) we can state

THEOREM 3.1. *If* $f\in C[0,1]$ *and*

$$\lim_{m\to\infty}a_m=0,\quad or\quad \lim_{m\to\infty}\frac{r_{m-2}(1)}{p_m(1)}=0, \tag{3.9}$$

then the sequence of polynomials of binomial type $(T_m^{\gamma,\delta}f)$, *where* $0\le\gamma\le\delta$, *converges to the function f, uniformly on the interval* $[0,1]$.

For proving it, we have to take into account the fact that the relations (3.3), (3.4) and (3.5)–(3.6), or (3.7)–(3.8) and the assumption (3.9) permit us to make use of the convergence criterion of Bohman-Korovkin.

4. EVALUATION OF THE ORDERS OF APPROXIMATION

6. Now we shall evaluate the order of approximation of a function $f\in C[0,1]$ by means of the operator T_m of Tiberiu Popoviciu.

For this purpose we can use two inequalities, which can be seen in H. H. Gonska and J. Meier [3], for the evaluation of the orders of approximation by using the first and second order moduli of smoothness ω_1 and ω_2.

If we approximate $f\in C[0,1]$ by means of T_mf, we obtain

$$|f(x)-(T_mf)(x)|\le\left[1+\frac{1}{\delta^2}T_m((t-x)^2;x)\right]\omega_1(f;\delta),\quad \delta\in\mathbb{R}_+.$$

By using the relations $T_me_0=e_0$, $T_me_1=e_1$ and (3.5)–(3.6), we get

$$|f(x)-(T_mf)(x)|\le\left[1+\frac{1}{\delta^2}\cdot\frac{x(1-x)}{m}(1+ma_m)\right]\omega_1(f;\delta).$$

If we choose $\delta=A\sqrt{\frac{x(1-x)}{m}}$ $(A>0)$, we obtain

$$|f(x)-(T_mf)(x)|\le\left[1+\frac{ma_m}{A^2}\right]\omega_1\left(f;\,A\sqrt{\frac{x(1-x)}{m}}\right).$$

This inequality permits us to see that T_mf is interpolatory at $x=0$ and $x=1$.

Now we can state

THEOREM 4.1. *If $f \in C[0,1]$ and we approximate f by $T_m f$, then we can give the following estimation of the order of approximation, by means of the first order modulus of continuity*

$$\|f - T_m f\| \leq \left(\frac{5}{4} + \frac{m}{4} a_m\right) \omega_1\left(f; \frac{1}{\sqrt{m}}\right).$$

In the case of Bernstein polynomials we have $a_m = 0$ and this inequality reduces to an inequality of Tiberiu Popoviciu [13], but with the coefficient $\frac{5}{4}$ obtained by G. G. Lorentz in his book [7].

7. If we use the second order modulus of smoothness, we obtain

$$|f(x) - T_m f)(x)| \leq \frac{3}{2}\left[1 + \frac{1}{2\delta^2} T_m((t-x)^2; x)\right] \omega_2(f; \delta),$$

or, more explicitly,

$$|f(x) - T_m f)(x)| \leq \frac{3}{2}\left[1 + \frac{1}{2\delta^2} \cdot \frac{x(1-x)}{m}(1+a_m)\right] \omega_2(f; \delta).$$

By choosing $\delta = A\sqrt{\frac{x(1-x)}{m}}$ $(A > 0)$, we obtain

$$|f(x) - T_m f)(x)| \leq \frac{3}{2}\left[1 + \frac{1 + ma_m}{2A^2}\right] \omega_2\left(f; \sqrt{\frac{x(1-x)}{m}}\right).$$

If we take into account that on [0, 1] we have $x(1-x) \leq \frac{1}{4}$ and we select $A = 2$, we can state

THEOREM 4.2. *By using the second order modulus of smoothness we can write the inequality*

$$\|f - T_m f\| \leq \frac{3}{16}(9 + ma_m)\, \omega_2\left(f; \frac{1}{\sqrt{m}}\right).$$

Since in the case of Bernstein polynomial we have $a_m = 0$, we arrive at an inequality of the form

$$\|f - B_m f\| \leq C\, \omega_2\left(f; \frac{1}{\sqrt{m}}\right),$$

where $C = \frac{27}{16} = 1{,}6875$. This value of the constant C was obtained in another way in [4].

5. A BINOMIAL TYPE OPERATOR DEPENDING ON FOUR PARAMETERS

8. If we consider the basic polynomials of binomial type $p_n^{\alpha,\beta}(x) = x(x+\alpha+n\beta)^{[n-1,-\alpha]}$, depending on two nonnegative parameters, we can write the identity

$$\sum_{k=0}^{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]} y(y+\alpha+(m-k)\beta)^{[m-1,-\alpha]} = $$
$$= (x+y)(x+y+\alpha+m\beta)^{[m-1,-\alpha]}.$$

If we set $y = 1 - x$ we obtain

$$(5.1)\quad \sum_{k=0}^{m} \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]} = $$
$$= (1+\alpha+m\beta)^{[m-1,-\alpha]}.$$

By starting from this equality we can construct a linear positive operator $Q_m^{\alpha,\beta,\gamma,\delta}$, depending on four parameters, defined, for any function $f \in C[0,1]$, by the formula

$$(5.2)\quad (Q_m^{\alpha,\beta,\gamma,\delta} f)(x) = \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) f\left(\frac{f+\gamma}{m+\delta}\right),$$

where $0 \leq \gamma \leq \delta$ and

$$(5.3)\quad (1+\alpha+m\beta)^{[m-1,-\alpha]} \cdot q_{m,k}^{\alpha,\beta}(x) = $$
$$= \binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

In the case $\alpha = \gamma = \delta = 0$ it reduces to an operator Q_m of Cheney-Sharma [1], while for $\beta = \gamma = \delta = 0$ one obtains an operator S_m^{α} of D. D. Stancu [18].

Since in the points $x = 0$ and $x = 1$ we have $f\left(\frac{\gamma}{m+\delta}\right)$, respectively $f\left(\frac{m+\gamma}{m+\delta}\right)$, we can see that the polynomial

$$(5.4)\quad (Q_m^{\alpha,\beta} f)(x) = \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) f\left(\frac{k}{m}\right)$$

has the property that

$$(Q_m^{\alpha,\beta} f)(0) = f(0), \qquad (Q_m^{\alpha,\beta} f)(1) = f(1).$$

Hence, it is expected that the approximation formula

(5.5) $$f(x) = (Q_m^{\alpha,\beta} f)(x) + (R_m^{\alpha,\beta} f)(x)$$

has the degree of exactness equal with one.

According to the identity (5.1), we can see that

(5.6) $$Q_m^{\alpha,\beta,\gamma,\delta} e_0 = e_0 .$$

In the case $e_1(x) = x$ we have

(5.7) $$(Q_m^{\alpha,\beta,\gamma,\delta} e_1)(x) = \frac{\gamma}{m+\delta} + \frac{m}{m+\delta}(Q_m^{\alpha,\beta} e_1)(x).$$

In accordance with (5.3) and (5.4), we can write

$$(1+\alpha+m\beta)^{[m-1,-\alpha]}(Q_m^{\alpha,\beta} e_1)(x) =$$

$$= \sum_{k=1}^{m} \frac{k}{m}\binom{m}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]} =$$

$$= \sum_{k=1}^{m} \binom{m-1}{k-1} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]}.$$

If we change the index of summation $k-1=j$, we obtain

(5.8) $$(1+\alpha+m\beta)^{[m-1,-\alpha]}(Q_m^{\alpha,\beta} e_1)(x) =$$

$$= x\sum_{j=0}^{m-1} \binom{m-1}{j}(x+\alpha+\beta+j\beta)^{[j,-\alpha]}(1-x)(1-x+(m-1-j)\beta)^{[m-2-j,-\alpha]}.$$

Now we consider an extension to factorial powers, with increment h, of an Abel combinatorial formula

(5.9) $$(u+v+n\beta)^{[n,h]} = \sum_{k=0}^{n}\binom{n}{k}(u+k\beta)^{[k,h]} v(v+(n-k)\beta-h)^{[n-1-k,h]}.$$

If we replace here $n=m-1, h=-\alpha, u=x+\alpha+\beta$ and $v=1-x$, we obtain

(5.10) $$(1+\alpha+m\beta)^{[m-1,-\alpha]} =$$

$$= \sum_{k=0}^{m-1}\binom{m-1}{k}(x+\alpha+\beta+k\beta)^{[k,-\alpha]}(1-x)(1-x+(m-1-k)\beta)^{[m-2-k,-\alpha]}.$$

By taking into account the identities (5.8) and (5.10), we obtain $Q_m^{\alpha,\beta} e_1 = e_1$. Consequently, the degree of exactness of formula (5.5) is indeed one.

If we replace this result in (5.7), we find that

(5.11) $$(Q_m^{\alpha,\beta,\gamma,\delta} e_1)(x) = \frac{mx+\gamma}{m+\delta} = x + \frac{\gamma-\delta x}{m+\delta}.$$

In the case of the monomial $e_2(x) = x^2$ we have

$$(Q_m^{\alpha,\beta,\gamma,\delta} e_2)(x) = \frac{1}{(m+\delta)^2}[\gamma^2 + 2m\gamma x + m^2(Q_m^{\alpha,\beta} e_2)(x)].$$

Next we can write

$$(Q_m^{\alpha,\beta} e_2)(x) = \frac{1}{m}\sum_{k=1}^{m}\left[\frac{k}{m} + \frac{k(k-1)}{m}\right] q_{m,k}^{\alpha,\beta}(x) = \frac{1}{m}(Q_m^{\alpha,\beta} e_1)(x) +$$

$$+\frac{1}{m}\sum_{k=2}^{m}\binom{m-1}{k-2} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+\alpha+(m-k)\beta)^{[m-1-k,-\alpha]} = \frac{x}{m} +$$

$$+\frac{m-1}{m}\sum_{j=0}^{m-2}\binom{m-2}{j}(x+\alpha+2\beta+j\beta)^{[j+1,-\alpha]}(1-x+\alpha+(m-2-j)\beta)^{[m-3-j,-\alpha]}.$$

If we take into account (5.6) and (5.11) and if we use again the extension to factorial powers of the Abel combinatorial formula, we can apply the Bohman-Korovkin convergence criterion and we can state

THEOREM 5.1. *If* $f \in C[0,1]$ *and the parameters* α *and* β *are nonnegative and depend on* m *such that* $\alpha = \alpha(m) \to 0, m\beta(m) \to 0,$ *when* $m \to \infty$, *then we have*

$$\lim_{m\to\infty} Q_m^{\alpha,\beta,\gamma,\delta} f = f,$$

uniformly on the interval [0, 1].

6. AN INTEGRAL REPRESENTATION OF THE REMAINDER IN THE APPROXIMATION FORMULA BY $Q_m^{\alpha,\beta}$

8. Since the degree of exactness of the approximation formula (5.5) is one, we can apply the Peano theorem in order to find an integral representation of the remainder term.

We can state

THEOREM 6.1. *Let x be any fixed point in* $[0,1]$. *If* $f\in C^2[0,1]$, *then the remainder of formula* (5.5) *can be represented under the following integral form*

$$(R_m^{\alpha,\beta}f)(x)=\int_0^1 G_m^{\alpha,\beta}(t;x)\,f''(t)\,dt, \tag{6.1}$$

where the Peano kernel $G_m^{\alpha,\beta}$ *is defined by the formula*

$$G_m^{\alpha,\beta}(t;x)=(R_m^{\alpha,\beta}\varphi_x)(t),\qquad \varphi_x(t)=\frac{1}{2}[x-t+|x-t|],$$

the operator acting on $\varphi_x(t)$ *as a function of x.*

We shall derive an explicit formula for the Peano kernel.

By starting from the equation

$$(R_m^{\alpha,\beta}\varphi_x)(t)=(x-t)_+-\sum_{k=0}^{\infty}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right)_+ \tag{6.2}$$

we shall be able to deduce an explicit expression for $G_m^{\alpha,\beta}$.

We can state

THEOREM 6.2. *Assuming that* $x\in\left[\frac{s-1}{m},\frac{s}{m}\right]$ $(1\le s\le m)$, *the Peano kernel* $G_m^{\alpha,\beta}$, *corresponding to the operator* $Q_m^{\alpha,\beta}$, *can be represented as follows:*

$$G_m^{\alpha,\beta}(t;x)=\begin{cases}-\sum\limits_{k=0}^{j-1}q_{m,k}^{\alpha,\beta}(x)\left(x-\frac{k}{m}\right) & \text{if } t\in\left[\frac{j-1}{m},\frac{j}{m}\right]\ (1\le j\le s-1)\\ -\sum\limits_{k=0}^{s-1}q_{m,k}^{\alpha,\beta}(x)\left(x-\frac{k}{m}\right) & \text{if } t\in\left(\frac{s-1}{m},x\right]\\ -\sum\limits_{k=s}^{m}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right) & \text{if } t\in\left(x,\frac{s}{m}\right]\\ -\sum\limits_{k=j}^{m}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right) & \text{if } t\in\left(\frac{j-1}{m},\frac{j}{m}\right]\ (s\le j\le m).\end{cases} \tag{6.3}$$

Proof. Since $x\in\left[\frac{s-1}{m},\frac{s}{m}\right]$, if we assume that $t\in\left[\frac{j-1}{m},\frac{j}{m}\right]$ $(1\le j\le s-1)$, we can write

$$G_m^{\alpha,\beta}(t;x)=x-t-\sum_{k\ge j}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right). \tag{6.4}$$

In the case $t\in\left[\frac{s-1}{m},x\right]$ we obtain

$$G_m^{\alpha,\beta}(t;x)=x-t-\sum_{k\ge s}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right) \tag{6.5}$$

and when $t\in\left[x,\frac{s}{m}\right]$ we get

$$G_m^{\alpha,\beta}(t;x)=-\sum_{k=s}^{m}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right).$$

When $t\in\left[\frac{j-1}{m},\frac{j}{m}\right]$ $(j>s)$ we have

$$G_m^{\alpha,\beta}(t;x)=-\sum_{k=j}^{m}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right).$$

Since the degree of exactness of formula (5.5) is one, by replacing in it $f(x)=x-t$, the corresponding remainder vanishes and we obtain

$$x-t=\sum_{k=0}^{m}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right)=\sum_{k=0}^{j-1}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right)+\sum_{k=j}^{m}q_{m,k}^{\alpha,\beta}(x)\left(\frac{k}{m}-t\right).$$

Consequently, if $x\in\left[\frac{s-1}{m},\frac{s}{m}\right]$ and $t\in\left[\frac{j-1}{m},\frac{j}{m}\right]$ we have

$$G_m^{\alpha,\beta}(t;x)=-\sum_{k=0}^{j-1}q_{m,k}^{\alpha,\beta}(x)\left(t-\frac{k}{m}\right),$$

while for $t\in\left(\frac{s-1}{m},x\right]$ we obtain

$$G_m^{\alpha,\beta}(t;x)=-\sum_{k=0}^{s-1}q_{m,k}^{\alpha,\beta}(x)\left(t-\frac{k}{m}\right).$$

9. We can also deduce a Cauchy type form for the remainder of the approximation formula (5.5).

THEOREM 6.3. *If* $f\in C^2[0,1]$, *the remainder of the approximation formula* (5.5) *can be represented under the form*

$$(R_m^{\alpha,\beta}f)(x)=\frac{1}{2}(R_m^{\alpha,\beta}e_2)(x)\,f''(\xi),\quad 0<\xi<1. \tag{6.6}$$

Proof. From (6.3) it is easy to see that the function $y = G_m^{\alpha,\beta}(x)$ represents a polygonal continuous line situated beneath the x-axis. Consequently, we can apply the mean value theorem to the integral occurring in (6.1) and we obtain

$$(6.7)\qquad (R_m^{\alpha,\beta} f)(x) = f''(\xi)\int_0^1 G_m^{\alpha,\beta}(t;x)\,dt, \quad \xi \in 0,1).$$

By inserting $f = e_2$ in the approximation formula

$$f(x) = (Q_m^{\alpha,\beta} f)(x) + f''(\xi)\int_0^1 G_m^{\alpha,\beta}(t;x)\,dt,$$

we obtain

$$\int_0^1 G_m^{\alpha,\beta}(t;x)\,dt = \frac{1}{2}[x^2 - (Q_m^{\alpha,\beta} e_2)(x)] = \frac{1}{2}(R_m^{\alpha,\beta} e_2)(x).$$

If we replace this result in (6.7), we arrive just to formula (6.6).

Since the polynomial $Q_m^{\alpha,\beta} f$ is interpolatory at both ends of [0, 1], it is clear that $(R_m^{\alpha,\beta} e_2)(x)$ contains the factor $x(x-1)$.

In the case $\beta = 0$ we find

$$(R_m^{\alpha} f)(x) = \frac{x(x-1)}{2m}\cdot\frac{1+\alpha m}{1+\alpha} f''(\xi),$$

given in [20] for the remainder of the approximation formula by the operator $S^{\alpha} = Q_m^{\alpha,0}$, introduced in 1968 in the papers [18] and [19] of the first author.

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