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ASYMPTOTIC APPROXIMATION WITH STANCU BETA OPERATORS

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Dedicated to Prof. Dr. D. D. Stancu on the occasion of his 70th birthday

Abstract. The concern of this paper is a beta type operator L_n recently introduced by D. D. Stancu. We present the complete asymptotic expansion for L_n as n tends to infinity. All coefficients of n^{-k} $(k = 1, 2, \ldots)$ are calculated explicitly in terms of Stirling numbers of the first and second kind. Moreover, we give an asymptotic expansion for L_n into a series of reciprocal factorials.

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1. INTRODUCTION

In his recent paper [17] D. D. Stancu introduced a new beta operator of second kind L_n associating, for $n \in \mathbb{N}$, with each $f \in M[0, \infty)$

$$\begin{aligned} & L_n(f; x) \equiv L_n(f(\cdot); x) \\ (1) \qquad & = \tfrac{1}{B(nx, n+1)} \int_0^\infty f(t) \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}}\, dt \qquad (x \in (0, \infty)), \end{aligned}$$

where $M[0, \infty)$ denotes the space of bounded and locally integrable functions on $[0, +\infty)$. Actually, L_n is applicable to functions of polynomial growth if we restrict ourselves to $n \geq n_0$ with $n_0 \in \mathbb{N}$ sufficiently great.

The operators (1) are constructed, roughly speaking, by applying a beta second kind transform $T_{p,q}$ to the Baskakov operators

$$(2) \qquad B_n(f; x) = \sum_{k=0}^{\infty} \tbinom{n+k-1}{k} x^k (1 + x)^{n+k} f\left(\tfrac{k}{n}\right) \qquad (x \in (0, \infty))$$

and then choosing $p = nx$ and $q = n + 1$.

The L_n are positive linear operators reproducing linear functions. It is mentioned in [17] that L_n is distinct from the other beta type operators considered earlier in the papers [14, 13, 18, 11, 6].

Furthermore, D. D. Stancu gave estimations for the degree of approximation by using the moduli of continuity of first and second orders. He established also an asymptotic formula of Voronovskaja–type

$$(3) \qquad \lim_{n\to\infty} n\big((L_n f)(x) - f(x)\big) = \tfrac{x(1+x)}{2} f''(x)$$

for all $f \in M[0,\infty)$ admitting a derivative of second order at x $(x > 0)$. The first result of this type was given by Voronovskaja [19] for the classical Bernstein polynomials and then generalized by Bernstein [8]. It should be worth noting that the right–hand term of Eq. (3) is the same as for the Baskakov operators (2).

The purpose of this paper is to present the complete asymptotic expansion for the operators (1) in the form

$$(L_nf)(x) \sim f(x) + \sum_{k=1}^{\infty} a_k(f;x)n^{-k} \qquad (n\to\infty), \tag{4}$$

provided f possesses derivatives of sufficiently high order at x $(x > 0)$. Formula (4) means that

$$(L_nf)(x) = f(x) + \sum_{k=1}^{q} a_k(f;x)n^{-k} + o(n^{-q}) \qquad (n\to\infty)$$

for all $q \in \mathbb{N}$. The Voronovskaja–type result (3) is the special case $q = 1$ with $a_1(f;x) = (1/2)x(1+x)f''(x)$. We give explicit expressions for all coefficients $a_k(f;x)$ $\;(k = 1,2,\ldots)$. The Stirling numbers of the first, resp. second, kind play an important role in our investigations.

It turns out that the representation of the moments L_ne_r with $e_r(x) = x^r$ $(r = 0,1,2,\ldots)$ requires an infinite series. In order to avoid this drawback we, finally, derive a closed expression for the moments by means of a finite series of reciprocal factorials. This leads to a further asymptotic expansion of the form

$$(L_nf)(x) \sim f(x) + \sum_{k=1}^{\infty} b_k(f;x)\frac{1}{n^{\underline{k}}} \qquad (n\to\infty), \tag{5}$$

where $n^{\underline{k}}$ denotes the falling factorial $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$, $n^{\underline{0}} = 1$.

We remark that in [1, 3, 2, 4] the author gave the analogous results for the operators of Meyer–König and Zeller, for the operators of Bleimann, Butzer and Hahn and the Stancu operators, respectively.

2. THE MOMENTS FOR THE STANCU BETA OPERATOR

A change of variable replacing t by $t/(1-t)$ in the definition (1) yields

$$(L_nf)(x) = \tfrac{1}{B(nx,n+1)} \int_0^1 f\left(\tfrac{t}{1-t}\right) t^{nx-1}(1-t)^n\, dt. \tag{6}$$

For the monomials e_r we get

$$(L_ne_r)(x) = \tfrac{B(nx+r,n+1-r)}{B(nx,n+1)} = \tfrac{(nx)^{\overline{r}}}{n^{\underline{r}}} \qquad (n > r-1), \tag{7}$$

where $n^{\overline{k}} = n(n+1)\cdots(n+k-1)$, $n^{\overline{0}} = 1$ is the rising factorial. Note that, in particular, $L_ne_r = e_r$ $(r = 0,1)$. For $r \geq 2$, we take advantage of the well–known identities

$$(nx)^{\overline{r}} = \sum_{j=0}^{r} (-1)^{r-j} S_r^j (nx)^j$$

and

$$\left[\left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\cdots\left(1-\tfrac{r-1}{n}\right)\right]^{-1} = \sum_{i=0}^{\infty} \sigma_{r-1+i}^{r-1} n^{-i} \qquad (n > r-1)$$

(see, e.g., [10, §60, Eq. (2), p.175]) which imply, by Eq. (7),

$$\begin{aligned}(L_n e_r)(x) &= n^{-r} \sum_{j=0}^{r} (-1)^j S_r^{r-j} (nx)^{r-j} \sum_{i=0}^{\infty} \sigma_{r-1+i}^{r-1} n^{-i} \\ &= x^r + \sum_{k=1}^{\infty} n^{-k} \sum_{j=0}^{\min\{k,r\}} (-1)^j S_r^{r-j} \sigma_{r-1+k-j}^{r-1} x^{r-j} \qquad (n > r-1). \qquad (8)\end{aligned}$$

Recall that the Stirling numbers S_j^i and σ_j^i of the first, resp. second, kind are defined by

$$x^{\underline{j}} = \sum_{i=0}^{j} S_j^i x^i \qquad \text{and} \qquad x^j = \sum_{i=0}^{j} \sigma_j^i x^{\underline{i}} \qquad (j \in \mathbb{N}_0).$$

Now we study the central moments

$$T_{n,s}(x) = L_n((\cdot - x)^s; x) \qquad (s = 0, 1, 2, \ldots).$$

Since L_n reproduces linear functions we have $T_{n,0}(x) = 1$ and $T_{n,1}(x) = 0$. Application of the binomial formula yields

$$T_{n,s}(x) = \sum_{r=0}^{s} \tbinom{s}{r} (-x)^{s-r} (L_n e_r)(x) \qquad (9)$$

and we get, by Eq. (8),

Proposition 1. *For each $x > 0$ and $s = 1, 2, \ldots$ the central moments $T_{n,s}$ of the operators L_n possess, for $n \geq s$, the representation*

$$T_{n,s}(x) = \sum_{k=1}^{\infty} n^{-k} \sum_{j=0}^{\min\{s,k\}} x^{s-j} \sum_{r=\max\{j,1\}}^{s} (-1)^{s-r+j} \tbinom{s}{r} S_r^{r-j} \sigma_{r-1+k-j}^{r-1}.$$

In order to derive as our main result the complete asymptotic expansion of the Stancu beta operators we use a general approximation theorem for positive linear operators due to Sikkema [16, Theorems 1 and 2].

As in [15] let $K^{[q]}(x)$ be the class of all functions $f \in M[0,\infty)$ which are q times differentiable at x ($x > 0$).

Theorem A. *For $q \in \mathbb{N}$ and fixed $x > 0$ let $A_n : K^{[2q]}(x) \to C(0,\infty)$ be a sequence of positive linear operators. If*

$$A_n((\cdot - x)^s; x) = \mathcal{O}(n^{-\lfloor (s+1)/2 \rfloor}) \quad (n \to \infty) \quad (s = 0, 1, \ldots, 2q+2) \tag{10}$$

then we have for each $f \in K^{[2q]}(x)$

$$A_n(f(\cdot); x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} A_n((\cdot - x)^s; x) + o(n^{-q}) \qquad (n \to \infty). \tag{11}$$

Furthermore, if $f \in K^{[2q+2]}(x)$, the term $o(n^{-q})$ in (11) *can be replaced by $\mathcal{O}(n^{-(q+1)})$.*

In order to apply Theorem A we have to show that assumption (10) is valid for the operators L_n. For this reason we prove

Proposition 2. *For each $x > 0$ and $s = 0, 1, 2, \ldots$ the central moments $T_{n,s}$ of the operators L_n satisfy*

$$T_{n,s}(x) = \mathcal{O}(n^{-\lfloor (s+1)/2 \rfloor}) \qquad (n \to \infty). \tag{12}$$

Proof. The case $s = 0$ is obvious. Let now $s \geq 1$ and put

$$Q(k, s, j) = \sum_{r=\max\{j,1\}}^{s} (-1)^r \tbinom{s}{r} S_r^{r-j} \sigma_{r-1+k-j}^{r-1}. \tag{13}$$

By Proposition 1, it is sufficient to show that

$$Q(k, s, j) = 0 \qquad (j = 0, \ldots, \min\{s, k\})$$

for all $k \in \mathbb{N}$ with $k < \lfloor (s+1)/2 \rfloor$. The Stirling numbers of the first, resp. second, kind occurring in Eq. (13) possess the representations

$$S_r^{r-j} = \sum_{\alpha=0}^{j} C_{j,j-\alpha} \tbinom{r}{j+\alpha} \qquad (j = 0, \ldots, r) \tag{14}$$

resp.

$$\sigma_{r-1+\ell}^{r-1} = \sum_{\beta=0}^{\ell} \bar{C}_{\ell,\ell-\beta} \tbinom{r-1+\ell}{\ell+\beta} \qquad (r \geq 1, \ell = 0, 1, 2, \ldots) \tag{15}$$

(see [10, p. 151, Eq. (5)], resp. [10, p. 171, Eq. (7)]). The coefficients $C_{j,i}$ and $\bar{C}_{j,i}$ are independent on r and satisfy certain partial difference equations whose general solutions are unknown ([10, p. 150]). Some closed expressions for $C_{j,i}$ and $\bar{C}_{j,i}$ can be found in [3].

In the case $j = 0$ we have

$$Q(k,s,j=0) = \sum_{r=1}^{s} (-1)^r \tbinom{s}{r} \sigma_{r-1+k}^{r-1}.$$

By Eq. (15), σ_{r-1+k}^{r-1} is a polynomial in r of degree $\leq 2k$ without constant summand if $k \geq 1$. Therefore, we have $Q(k,s,j=0) = 0$ if $0 < 2k < s$.

On the other hand, if $j \geq 1$ we insert (14) into Eq. (13) and get

$$\begin{aligned} Q(k,s,j) &= \sum_{\alpha=0}^{j} C_{j,j-\alpha} \sum_{r=j}^{s} (-1)^r \tbinom{s}{r} \tbinom{r}{j+\alpha} \sigma_{r-1+k-j}^{r-1} \\ &= \sum_{\alpha=0}^{j} C_{j,j-\alpha} \tfrac{s^{\underline{j}}}{(j+\alpha)!} \sum_{r=0}^{s-j} (-1)^{r+j} \tbinom{s-j}{r} r^{\underline{\alpha}} \sigma_{r-1+k}^{r-1+j}. \end{aligned}$$

Since $r^{\underline{\alpha}} \sigma_{r-1+k}^{r-1+j}$ is a polynomial in r of degree $\leq j+2(k-j)$ we have $Q(k,s,j) = 0$ if $j + 2(k-j) < s - j$, i.e., if $2k < s$. Note that we only have to consider the case $s - j > 0$. This completes the proof of Proposition 2. □

3. THE COMPLETE ASYMPTOTIC EXPANSION

Combining Proposition 1, Theorem A and Proposition 2 we get our first main result.

THEOREM 1. *Let $x > 0, q \in \mathbb{N}$ and $f \in K^{[2q]}(x)$. Then, the Stancu beta operators possess the asymptotic expansion*

$$(L_n f)(x) = f(x) + \sum_{k=1}^{q} a_k(f;x) n^{-k} + o(n^{-q}) \qquad (n \to \infty), \tag{16}$$

where the coefficients $a_k(f;x)$ are given by

$$a_k(f;x) = \sum_{s=2}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{\min\{k,s\}} x^{s-j} \sum_{r=\max\{j,1\}}^{s} (-1)^{s-r+j} \tbinom{s}{r} S_r^{r-j} \sigma_{r-1+k-j}^{r-1}. \tag{17}$$

For the convenience of the reader we calculate the initial coefficients. As usual, we put $X = x(1+x)$ (cf. [9, Theorem 1.1, p. 303]) and $X' = 1 + 2x$.

$$a_1(f;x) = \tfrac{1}{2} X f''(x),$$

$$\begin{aligned}
a_2(f;x) &= \tfrac{1}{2}Xf^{(2)}(x) + \tfrac{1}{3}XX'f^{(3)}(x) + \tfrac{1}{8}X^2f^{(4)}(x),\\
a_3(f;x) &= \tfrac{1}{2}Xf^{(2)}(x) + XX'f^{(3)}(x)\\
&\quad + \tfrac{1}{4}X(6X+1)f^{(4)}(x) + \tfrac{1}{6}X^2X'f^{(5)}(x) + \tfrac{1}{48}X^3f^{(6)}(x),\\
a_4(f;x) &= \tfrac{1}{2}Xf^{(2)}(x) + \tfrac{7}{3}XX'f^{(3)}(x) + \tfrac{1}{8}X(61X+12)f^{(4)}(x)\\
&\quad + \tfrac{1}{15}XX'(31X+3)f^{(5)}(x) + \tfrac{1}{144}X^2(131X+26)f^{(6)}(x)\\
&\quad + \tfrac{1}{24}X^3X'f^{(7)}(x) + \tfrac{1}{384}X^4f^{(8)}(x).
\end{aligned}$$

We mention that, in particular, the coefficient $a_3(f;x)$ possesses the concise form

$$a_3(f;x) = \tfrac{1}{48}X\left(\tfrac{d}{dx}\right)^{(4)}\Big(X^2f^{(2)}(x)\Big).$$

For the sake of comparison we give without proof the initial terms of the asymptotic expansion of the Baskakov operators for $x>0$ and $f\in K^{[6]}(x)$:

$$\begin{aligned}
B_n(f;x) =& f(x) + \tfrac{1}{2n}Xf''(x) + \tfrac{1}{24n^2}\Big(4XX'f^{(3)}(x) + 3X^2f^{(4)}(x)\Big)\\
&+ \tfrac{1}{48n^3}\Big(2X(6X+1)f^{(4)}(x) + 4X^2X'f^{(5)}(x) + X^3f^{(6)}(x)\Big)\\
&+ o(n^{-3}) \qquad (n\to\infty).
\end{aligned}$$

4. THE COMPLETE ASYMPTOTIC EXPANSION INTO A SERIES OF RECIPROCAL FACTORIALS

Direct computation of the moments $(L_ne_r)(x) = (nx)^{\overline{r}}/n^{\underline{r}} \quad (n > r-1)$ as given in Eq. (7) yields

$$\begin{aligned}
(L_ne_1)(x) =& x,\\
(L_ne_2)(x) =& x^2 + \tfrac{x^2+x}{n-1},\\
(L_ne_3)(x) =& x^3 + 3\tfrac{x^3+x^2}{n-1} + 2\tfrac{2x^3+3x^2+x}{(n-1)(n-2)}.
\end{aligned}$$

These expressions suggest to study the expansion of the operators L_n into a series of reciprocal factorials. Proposition 3 exhibits the general expression for the moments.

PROPOSITION 3. *For $r = 0, 1, 2, \ldots$ the moments $L_n e_r$ of the operators L_n possess the representation*

(18)

$$(L_{n+1}e_r)(x) = x^r + \sum_{k=1}^{r-1} \frac{(-1)^k}{n^{\underline{k}}} \sum_{j=0}^{k} x^{r-j} \sum_{i=0}^{k-j} (-1)^i \binom{r-1-j}{i} S_r^{r-j} \sigma_{r-1-j-i}^{r-1-k} (r-1)^i,$$

$n \geq r$, where the sum $\sum\limits_{k=1}^{r-1}$ is to be read as 0 in the case $r \leq 1$.

Proof. The case $r = 0$ is obvious. Let $r \in \mathbb{N}$ and $n \geq r$. We take advantage of the identities

$$(nx)^{\overline{r}} = \sum_{j=1}^{r} (-1)^{r-j} S_r^j (nx)^j,$$

$$n^j = n(n-r+1+r-1)^{j-1} = n \sum_{i=0}^{j-1} (-1)^i \binom{j-1}{i} (r-1)^{j-1-i} [-(n-r+1)]^i,$$

$$[-(n-r+1)]^i = \sum_{k=0}^{i} \sigma_i^k [-(n-r+1)]^{\underline{k}} = \sum_{k=0}^{i} \sigma_i^k (-1)^k (n-r+k)^{\underline{k}},$$

and therefore

$$\frac{n^j}{n^{\underline{r}}} = n \sum_{i=0}^{j-1} \sum_{k=0}^{i} (-1)^{i+k} \binom{j-1}{i} \sigma_i^k (r-1)^{j-1-i} \frac{1}{n^{\underline{r-k}}}.$$

Combining these formulae with Eq. (7) we receive

$$\begin{aligned}
(L_n e_r)(x) &= \frac{(nx)^{\overline{r}}}{n^{\underline{r}}} \\
&= \sum_{j=0}^{r-1} (-1)^{r-j-1} S_r^{j+1} x^{j+1} \sum_{i=0}^{j} \sum_{k=0}^{i} (-1)^{i+k} \binom{j}{i} \sigma_i^k (r-1)^{j-i} \frac{n}{n^{\underline{r-k}}} \\
&= \sum_{k=0}^{r-1} \frac{(-1)^k}{(n-1)^{\underline{r-1-k}}} \sum_{j=k}^{r-1} x^{j+1} \sum_{i=k}^{j} (-1)^{r+i-j-1} \binom{j}{i} S_r^{j+1} \sigma_i^k (r-1)^{j-i}
\end{aligned}$$

which yields Eq. (18) after some manipulations. □

By Eq. (9) and Proposition 3, we get for the central moments $T_{n,s}$ of the operators L_n, for $n+1 \geq s \geq 1$,

$$T_{n+1,s}(x) = \\ = \sum_{k=1}^{s-1} \frac{(-1)^k}{n^{\underline{k}}} \sum_{j=0}^{k} x^{s-j} \sum_{r=k+1}^{s} (-1)^{s-r} \binom{s}{r} \sum_{i=0}^{k-j} (-1)^i \binom{r-1-j}{i} S_r^{r-j} \sigma_{r-1-j-i}^{r-1-k} (r-1)^i.$$

Thus, we receive as our second main result, by Theorem A and Proposition 2

THEOREM 2. *Let $x > 0$, $q \in \mathbb{N}$ and $f \in K^{[2q]}(x)$. Then, the Stancu beta operators possess an asymptotic expansion into a reciprocal factorial series*

$$(L_n f)(x) = f(x) + \sum_{k=1}^{q} \frac{b_k(f;x)}{(n-1)^{\underline{k}}} + o(n^{-q}) \qquad (n \to \infty), \tag{19}$$

where the coefficients $b_k(f;x)$ are given by

(20)

$$b_k(f;x) = \\ = (-1)^k \sum_{s=k+1}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} x^{s-j} \sum_{r=k+1}^{s} (-1)^{s-r} \binom{s}{r} \sum_{i=0}^{k-j} (-1)^i \binom{r-1-j}{i} S_r^{r-j} \sigma_{r-1-j-i}^{r-1-k} (r-1)^i.$$

For the convenience of the reader we calculate the initial coefficients:

$$\begin{aligned}
b_1(f;x) &= \tfrac{1}{2} X f''(x), \\
b_2(f;x) &= \tfrac{1}{3} X X' f^{(3)}(x) + \tfrac{1}{8} X^2 f^{(4)}(x), \\
b_3(f;x) &= \tfrac{1}{8} X(9X+2) f^{(4)}(x) + \tfrac{1}{6} X^2 X' f^{(5)}(x) + \tfrac{1}{48} X^3 f^{(6)}(x), \\
b_4(f;x) &= \tfrac{1}{15} X X'(16X+3) f^{(5)}(x) + \tfrac{1}{144} X^2 (113X+26) f^{(6)}(x) \\
&\quad + \tfrac{1}{24} X^3 X' f^{(7)}(x) + \tfrac{1}{384} X^4 f^{(8)}(x), \\
b_5(f;x) &= \left(\tfrac{1}{6} X + \tfrac{125}{144} X^2 (5X+2)\right) f^{(6)}(x) + \tfrac{1}{120} X^2 X'(109X+22) f^{(7)}(x) \\
&\quad + \tfrac{1}{576} X^3 (145X+34) f^{(8)}(x) + \tfrac{1}{144} X^4 X' f^{(9)}(x) + \tfrac{1}{3840} X^5 f^{(10)}(x).
\end{aligned}$$

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