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LINEAR COMBINATIONS OF D. D. STANCU POLYNOMIALS

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1. INTRODUCTION

In the paper [10], D. D. Stancu has introduced and investigated a linear operator $P_m^{\langle\alpha\rangle}$ which maps the space $C[0, 1]$ into itself and is defined by

$$(P_m^{\langle\alpha\rangle} f)(x) = \sum_{k=0}^{m} w_{m,k}(x;\alpha) f\left(\frac{k}{m}\right),$$

where

$$w_{m,k}(x;\alpha) = \binom{m}{k} \frac{\prod_{\nu=0}^{k-1} (x+\nu\alpha) \prod_{\mu=0}^{m-k-1} (1-x+\mu\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{m-1}\alpha)},$$

α being a parameter which may depend only on the natural number m. If α is non-negative, then these operators preserve the positivity of the function f. We remind that $w_{m,k}(\cdot;\alpha)$ are known as "the fundamental polynomials of Stancu" and, by using the factorial powers we can write

$$w_{m,k}(x;\alpha) = \binom{m}{k} x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]} / 1^{[m,-\alpha]}.$$

It is easy to check the following statements: $P_m^{\langle\alpha\rangle} f$ interpolates the function f at the extremities of the interval [0, 1]; for $\alpha = -1/m$, the operator becomes the Lagrange interpolating polynomial corresponding to the equally spaced nodes k/m, $k = \overline{0, m}$; for $\alpha = 0$, $P_m^{\langle 0\rangle}$ coincides with the classical Bernstein operator.

In the next section we shall present some basic results connected with these operators, which have been established by D. D. Stancu and several other authors. In the last part of the paper we shall deal with certain linear combinations of

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Stancu polynomials which under definite conditions, approximate a function more closely than the above polynomials.

2. A SURVEY OF THE MAIN RESULTS CONCERNING THE OPERATOR $P_m^{\langle\alpha\rangle}$

Firstly, we remark that, by using a probability distribution which is connected with the Markov-Pólya urn scheme, a probabilistic interpretation of the polynomials $P_m^{\langle\alpha\rangle} f$ has been given in [11].

In [10], Stancu has established a relation between two consecutive terms of the sequence $(P_m^{\langle\alpha\rangle} f), \alpha \geq 0$, which involves the second order divided differences

$$(P_{m+1}^{\langle\alpha\rangle} f)(x) - (P_m^{\langle\alpha\rangle} f)(x) =$$

$$= -\frac{1}{m(m+1)} \sum_{\nu=0}^{m-1} \frac{(x+\nu\alpha)(1-x+\overline{m-\nu-1}\alpha)}{(1+m\alpha)(1+m-1\alpha)} w_{m-1,\nu}(x;\alpha) \left[\frac{\nu}{m}, \frac{\nu+1}{m+1}, \frac{\nu+1}{m}; f\right].$$

This representation has led to the study of the monotonicity properties, as follows: if f is convex (concave) of first order on $[0,1]$, then the sequence $(P_m^{\langle\alpha\rangle} f)$ is decreasing (increasing) on $[0,1]$. By using the beta function, the author has proved that for $\alpha > 0$ and every $x \in (0,1)$ the following identity

$$w_{m,k}(x;\alpha) = \binom{m}{k} \frac{B(x\alpha^{-1}+k, (1-x)\alpha^{-1}+m-k)}{B(x\alpha^{-1}, (1-x)\alpha^{-1})}$$

holds. Consequently, $P_m^{\langle\alpha\rangle}$ can be represented by means of the Bernstein operator B_m according to the formula

$$(1) \quad (P_m^{\langle\alpha\rangle} f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1} (1-t)^{\frac{1-x}{\alpha}-1} (B_m f)(t)\, dt, \quad x \in (0,1).$$

Concerning the estimate of the order of approximation, D. D. Stancu has proved

i) if $f \in C[0,1]$, then $|f(x) - (P_m^{\langle\alpha\rangle} f)(x)| \leq \frac{3}{2}\omega(f,\delta)$,

ii) if $f \in C^1[0,1]$, then $|f(x) - (P_m^{\langle\alpha\rangle} f)(x)| \leq \frac{3}{4}\delta\omega(f',\delta)$,

here $\delta = \{(1+\alpha m)/(m+\alpha m)\}^{1/2}$.

Further, Gonska and Meier [3] have improved the values of the constants finding 5/4, respectively 5/8.

Furthermore, Stancu has found various representations of the remainder term $R_m^{\langle\alpha\rangle} f = f - P_m^{\langle\alpha\rangle} f$. We quote here the following formulae [14]:

$$(R_m^{\langle\alpha\rangle} f)(x) = -\sum_{k=0}^{m-1} \frac{(x+k\alpha)(1-x+\overline{m-1-k}\alpha)}{m(1+m-1\alpha)} w_{m-1,k}(x;\alpha)\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right]$$

and

$$(R_m^{\langle\alpha\rangle} f)(x) = -\frac{x(1-x)}{m}\frac{1+m\alpha}{1+\alpha}[\xi_{m,1}, \xi_{m,2}, \xi_{m,3}; f],$$

where $\xi_{m,1}, \xi_{m,2}, \xi_{m,3}$ are distinct points on $[0,1]$. If we assume that $f \in C^2(0,1)$, we can obtain the relation

$$(R_m^{\langle\alpha\rangle} f)(x) = -\frac{x(1-x)}{m}\frac{1+m\alpha}{1+\alpha} f''(\xi_m), \quad \xi_m \in (0,1),$$

or an integral representation

$$(R_m^{\langle\alpha\rangle} f)(x) = \int_0^1 (R_m^{\langle\alpha\rangle} \psi_x)(t) f''(t)\, dt, \quad \psi_x(t) = (x-t)_+.$$

Under the hypotheses $0 \leq \alpha = \alpha(m) \to 0$ as $m \to \infty$ and f possesses a second derivative at a point $x \in [0,1]$, an asymptotic estimate of Voronovskaja type was given

$$(R_m^{\langle\alpha\rangle} f)(x) = -\frac{1+m\alpha}{1+\alpha}\frac{x(1-x)}{2m} f''(x) + \frac{\varepsilon_m^{\langle\alpha\rangle}(x)}{m},$$

where $\varepsilon_m^{\langle\alpha\rangle}(x)$ tends to 0 when m tends to ∞. According to paper [13, p. 36], the result is also valid if $\alpha = \alpha(m)$ is a non positive number so that $-m\alpha(m) \leq \varepsilon$, where $0 \leq \varepsilon < 1/2$ and $x \in [\varepsilon, 1-\varepsilon]$. If we further assume that $m\alpha(m) \to a, a > -1/2$, then we can write $\lim_{m\to\infty} m[f(x) - (P_m^{\langle\alpha\rangle} f)(x)] = -(1/2)(1+a)\,x(1-x)\,f''(x)$.

Under the hypothesis that f has the derivative of order $2p$ at x, Mühlbach [8] obtained a generalization of this result.

Mastroianni and Occorsio [5] studied the sequence $(P_m^{\langle\alpha\rangle} f)^{(p)}$ of derivatives of order p $(0 \leq p < m)$ of $(P_m^{\langle\alpha\rangle} f)$. For $f \in C^p[0,1]$ they proved $\lim_{m\to\infty} (P_m^{\langle\alpha\rangle} f)^{(p)}(x) = f^{(p)}(x)$, where $x \in [0,1]$ and $\alpha = \alpha(m) = o(m^{-1})$. The same authors [6] introduced and investigated the iterates of Stancu operator, defined as follows

$$(P_m^{\langle\alpha\rangle})^0 = I, \quad (P_m^{\langle\alpha\rangle})^1 = P_m^{\langle\alpha\rangle}, \quad (P_m^{\langle\alpha\rangle})^j = P_m^{\langle\alpha\rangle}(P_m^{\langle\alpha\rangle})^{j-1}, \quad j > 1.$$

They proved the following limiting relation

$$\lim_{j\to\infty} (P_m^{\langle\alpha\rangle})^j (f;x) = f(0) + (f(1) - f(0))\,x,$$

uniformly on $[0,1]$, for any $\alpha \geq 0$.

Finally, we mention two important papers: [1] and [2]. In the former, Della Vecchia elaborated a well-informed synthesis of the principal results obtained in the theory of uniform approximation of continuous functions by means of various classes of linear positive operators of D. D. Stancu, and in the latter Di Lorenzo and Occorsio achieved a systematic presentation of Stancu polynomials $P_m^{\langle\alpha\rangle}$.

3. A NEW RESULT

In [6] it was treated a linear combination of the iterates of Stancu polynomials defined by the operatorial formula $S_{m,k}^{\alpha} := I - (I - P_m^{\langle\alpha\rangle})^k = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} (P_m^{\langle\alpha\rangle})^i$, where I represents the identity operator. $S_{m,k}^{\alpha}$ gives a better approximation than $P_m^{\langle\alpha\rangle} = S_{m,1}^{\alpha}$ for sufficiently smooth functions. For the same operator $P_m^{\langle\alpha\rangle}$ we study here another type of linear combinations based on the works of Rathore [9] and May [7].

In what follows, we denote $e_j : [0,1] \to \mathbb{R}$, $e_j(x) = x^j$, $j \geq 0$. Let $d_0, d_1, \ldots, d_k$ be $k+1$ arbitrary, fixed and distinct positive integers. We define

(2) $$c(0,0) = 1 \quad \text{and} \quad c(i,k) = d_i^k \prod_{\substack{j=0 \\ j\neq i}}^{k} (d_i - d_j)^{-1}, \quad k \neq 0.$$

These coefficients enjoy the properties

(3) $$\sum_{i=0}^{k} c(i,k) = 1; \quad \sum_{i=0}^{k} c(i,k)\, d_i^{-m} = 0, \quad 1 \leq m \leq k.$$

Indeed, let us take $L_k f$ the Lagrange interpolating polynomial corresponding to the function f and the nodes d_i^{-1}, $i = \overline{0,k}$,

$$(L_k f)(x) = \sum_{i=0}^{k} \frac{u(x)}{(x - d_i^{-1}) \frac{du}{dx}(d_i^{-1})} f(d_i^{-1}),$$

where $u(x) = (x - d_0^{-1})(x - d_1^{-1}) \ldots (x - d_k^{-1})$. It is clear that for any $m \leq k$, $(L_k e_m)(x) = e_m(x)$. For $x = 0$, this implies $(L_k e_0)(0) = 1$ and $(L_k e_m)(0) = 0$ for $1 \leq m \leq k$. On the other hand, we can write

$$(L_k e_m)(0) = \sum_{i=0}^{k} \frac{(-1)^k d_i^{k-m}}{(d_0 - d_i) \ldots / \ldots (d_k - d_i)} = \sum_{i=0}^{k} c(i,k)\, d_i^{-m},$$

which leads us to (3).

By using the coefficients defined by (2), we are able to form a linear combination of Stancu operators as follows:

(4) $$(D_{m,k}^{\alpha} f)(x) = \sum_{i=0}^{k} c(i,k) (P_{d_i m}^{\langle\alpha\rangle} f)(x).$$

For $d_0 = 1$, one obtains $D_{m,0}^{\alpha} = P_m^{\langle\alpha\rangle}$. Also, we have $(D_{m,k}^{\alpha} f)(x_0) = f(x_0)$ for $x_0 = 0$ and $x_0 = 1$..

THEOREM 1. *The following identities*

(5) $$D_{m,k}^{\alpha} e_0 = e_0, \quad D_{m,k}^{\alpha} e_1 = e_1, \quad D_{m,k}^{\alpha} e_2 = \frac{\alpha}{\alpha+1} e_1 + \frac{1}{\alpha+1} e_2$$

hold.

Proof. Taking into account the next identities [10, Lemma 4.1]

$$(P_m^{\langle\alpha\rangle} e_0)(x) = 1, \quad (P_m^{\langle\alpha\rangle} e_1)(x) = x,$$

$$(P_m^{\langle\alpha\rangle} e_2)(x) = \frac{1}{1+\alpha} \left(\frac{x(1-x)}{m} + x(x+\alpha) \right),$$

and the relation (4), we easily obtain the desired result.

At this point, we introduce the s-th order central moments of the operator $P_m^{\langle\alpha\rangle}$, that is $\mu_{m,s}(\alpha;x) = (P_m^{\langle\alpha\rangle} \varphi_x^s)(x)$, where $\varphi_x = e_1 - x e_0$.

It should be mentioned that the recurrence relation for the central moments of the Markov-Pólya distribution has been established by D. D. Stancu [12].

LEMMA 1. *Let us assume that* $0 < \alpha$. *The following identity*

(6) $$m^s \mu_{m,s}(\alpha;x) = \sum_{i=0}^{[s/2]} \theta_{s,i}(\alpha;x)\, m^i, \quad m \geq 1,\ s \geq 0,\ x \in [0,1],$$

holds, where $\theta_{s,i}(\alpha;x)$ *are polynomials in* x *of degree less than or equal to* s.

Proof. Knowing that the operator $P_m^{\langle\alpha\rangle}$ is interpolatory at both sides of the interval $[0,1]$, (6) becomes true for $x = 0$ and $x = 1$.

Next, we consider only $0<x<1$. We recall an old result concerning the Bernstein operators (see [4, § 1.5, pp. 13 – 14]). If we consider the expressions $T_{m,s}(x)=\sum_{i=0}^{m}\binom{m}{i}x^i(1-x)^{m-i}(i-mx)^s$, then for a fixed s, $T_{m,s}(x)$ is a polynomial in x and m; in x of degree $\le s$, in m of degree $[s/2]$. Thus we can write $T_{m,s}(x)=\sum_{i=0}^{[s/2]}\phi_{s,i}(x)\,m^i$, where $\phi_{s,i}$ are polynomials in x of degree $\le s$, independent of m. By using the relation (1), we have

$$m^s\mu_{m,s}(\alpha;x)=\frac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-s}{\alpha}-1}T_{m,s}(t)\,dt=$$

$$=\frac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\sum_{i=0}^{[s/2]}m^i\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\phi_{s,i}(t)\,dt.$$

For any integer $k\ge 1$, we can write

$$\frac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}t^k\,dt=\frac{B\left(\frac{x}{\alpha}+k,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}=\prod_{j=0}^{k-1}\left(\frac{x+j\alpha}{1+j\alpha}\right),$$

which represents a polynomial in x of degree k. The above identities complete the proof of our lemma.

We further assume that $0<\alpha=\alpha(m)\to 0$ as $m\to\infty$. Keeping the notations used in the above proof, for $s\ge 1$, it results $\phi_{s,i}(0)=0$ and we can put $\phi_{s,i}(t)=\sum_{k=1}^{s}a_{s,i,k}t^k$. The coefficients $a_{s,i,k}$ are independent of α. For $x\in[0,1]$ and $0<\alpha$ we have

$$|\theta_{s,i}(\alpha;x)|\le\sum_{k=1}^{s}|a_{s,i,k}|\prod_{j=0}^{k-1}\frac{x+j\alpha}{1+j\alpha}\le s b_{s,i},$$

where $b_{s,i}=\max_{1\le k\le s}|a_{s,i,k}|$. Now, we can state the following result:

LEMMA 2. *If $0<\alpha(m)\to 0$ as $m\to\infty$, then there exists a constant $\beta_{s,i}$ such that*

(7) $$|\theta_{s,i}(\alpha;x)|\le\beta_{s,i},\quad x\in[0,1].$$

THEOREM 2. *Let us assume that $0<\alpha(m)\to 0$ as $m\to\infty$. For any $s\ge 1$ one has*

$$\left\|D^{\alpha}_{m,k}\varphi^s_x\right\|=\mathbf{O}(m^{-(k+1)}),\ m\to\infty,$$

where $\|\cdot\|$ stands for the sup norm on $C[0,1]$.

Proof. By using relations (4) and (6), we get

$$(D^{\alpha}_{m,k}\varphi^s_x)(x)=\sum_{j=0}^{k}c(j,k)\,\mu_{d_jm,s}(\alpha;x)=$$

$$=\sum_{j=0}^{k}c(j,k)(d_jm)^{-s}\sum_{i=0}^{[s/2]}\theta_{s,i}(\alpha;x)(d_jm)^i=$$

$$=\frac{1}{m^{k+1}}\sum_{i=0}^{[s/2]}\frac{\theta_{s,i}(\alpha;x)}{m^{s-i-(k+1)}}\sum_{j=0}^{k}c(j,k)\frac{1}{d_j^{s-i}}.$$

According to (3), we have $\sum_{j=0}^{k}c(j,k)\,d^{-(s-i)}=0$ for $s-i=1,2,\ldots,k$. We shall consider $s-i\ge k+1$. Taking into account (7), we further write

$$|(D^{\alpha}_{m,k}\varphi^s_x)(x)|\le\frac{1}{m^{k+1}}\sum_{i=0}^{[s/2]}|\theta_{s,i}(\alpha;x)|\sum_{j=0}^{k}|c(j,k)|\frac{1}{d_j^{s-i}}\le\frac{\gamma}{m^{k+1}},$$

where γ is a constant that depends on the numbers $s,d_0,d_1,\ldots,d_k$ but is independent of m. The result follows.

THEOREM 3. *Let us assume that $0<\alpha(m)\to 0$ as $m\to\infty$. If f is bounded on $[0,1]$ and is $2k+2$ times differentiable at some point $x\in[0,1]$, then*

$$|(D^{\alpha}_{m,k}f)(x)-f(x)|\le C_k m^{-(k+1)},$$

where C_k is a constant that depends on k.

Proof. By using the Taylor expansion to f, we can write for all $t\in[0,1]$

$$f(t)=f(x)+\sum_{r=1}^{2k+2}\frac{f^{(r)}(x)}{r!}(t-x)^r+(t-x)^{2k+2}h(t-x),$$

where h is bounded and $\lim_{u\to 0}h(u)=0$. By virtue of the linearity of the operator $D^{\alpha}_{m,k}$ and by relations (4) and (5) we obtain

$$(D^{\alpha}_{m,k}f)(x)-f(x)=\sum_{r=1}^{2k+2}\frac{f^{(r)}(x)}{r!}(D^{\alpha}_{m,k}\varphi^r_x)(x)+\sum_{j=0}^{k}c(j,k)(P^{\langle\alpha\rangle}_{d_jm}R_{2k+2,x})(x),$$

where $R_{2k+2,x}=\varphi_x^{2k+2}h(\cdot-x)$. Choosing $s=2k+2$ in (6), from the boundness of h, Lemma 2 and the Cauchy inequality it results that there exists a constant γ which depends on k and d_j such that

$$(P_{d_jm}^{\langle\alpha\rangle}R_{2k+2,x})(x)\le\gamma m^{-(k+1)}.$$

Theorem 3 follows from (3) and Theorem 2.

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