REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome XXVII, N° 1, 1998, pp. 23–31

ON THE MONOTONE CONVERGENCE OF AN EULER-CHEBYSHEFF-TYPE METHOD IN PARTIALLY ORDERED TOPOLOGICAL SPACES

IOANNIS K. ARGYROS

1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution x^* of the nonlinear operator equation

(1) $$F(x)=0$$

in a linear space E_1, where F is defined on a convex subset D of D_1 with values in a linear space E_2.

We have recently shown that if E_1 and E_2 are Banach spaces, then under standard Newton-Kantorovich hypotheses the Euler-Chebysheff-type method of the form

(2) $$y_n = x_n - [x_n, x_n]^{-1} F(x_n)$$

(3) $$x_{n+1} = y_n - [x_n, x_n]^{-1}([x_n, y_n]-[x_n, x_n])(y_n - x_n) \quad x_0 \in D \quad (n \ge 0)$$

converges with order almost three to a locally unique solution $x^* \in D$ of equation (1). Here $[x, y]$ denotes a divided difference of order one, which is a linear operator.

We introduce and study the monotone convergence of the iterations $\{v_n\}$ and $\{x_n\}$ $(n \ge 0)$ given by

(4) $$F(v_n)+[x_n, x_n](w_n - v_n)=0$$

(5) $$F(x_n)+[x_n, x_n](y_n - x_n)=0$$

(6) $$([x_n, y_n]-[x_n, x_n])(w_n - v_n)+[x_n, x_n](v_{n+1} - w_n)=0$$

and

(7) $$([x_n, y_n]-[x_n, x_n])(y_n - x_n)+[x_n, x_n](x_{n+1} - y_n)=0$$

to approximate a solution x^* of equation (1).

(1991) AMS (MOS) Subject Classification Codes. 65H10, 65J15, 49D15, 47H17.

The Euler-Chebysheff method (or the method of tangent parabolas) converges with order three ([5], [6]). However, with the exception of some special cases, this method has no practical value in a Banach space setting because it requires an evaluation of the second Fréchet-derivative at each step (which means a number of function evaluations proportional to the cube of the dimension of the space). Discretized versions of this method were considered by Ul'm [8] and Potra [7]. Ul'm used divided differences of order one and two, whereas Potra used divided differences of order one only. However, Potra used hypotheses on divided diffe-rences of order two in his convergence theorem [7, p. 91]. The order of convergence of his iteration is 1.839 … . The order of convergence of our iterations is almost three. Moreover, we use hypotheses on divided differences of order one only.

2. MONOTONE CONVERGENCE

We shall assume that the reader is familiar with the meaning of a divided difference of order one and the notion of a partially ordered topological space (POTL) ([1], [2], [7], [9]). Moreover, from now on we shall assume that E_1 and E_2 are POTL-spaces.

We can now state the main result.

THEOREM 1. *Let F be a nonlinear operator defined on a convex subset D of a regular POTL-space E_1 with values in a POTL-space E_2. Let v_0 and x_0 be two points of D such that*

(8) $$v_0 \le x_0$$

(9) $$F(v_0) \le 0 \le F(x_0).$$

Suppose that F has a divided difference of order one on $D_0 = \langle v_0, x_0 \rangle = \{x \in E_1 \,|\, v_0 \le x \le x_0\} \subseteq D$ satisfying

(10) $A_0 = [x_0, x_0]$ *has a continuous nonnegative left subinverse B_0,*

(11) $$[x_0, y] \ge 0 \quad \text{for all} \quad v_0 \le y \le x_0,$$

(12) $$[x, v] - [x, y] \le 0 \quad \text{if} \quad v \le y$$

and

(13) $[z, w] + [w, q] - [z, z] - [v, z] \ge 0$ *if* $v \le w \le z$ *for some* $q \in \langle v, z \rangle$.

Then there exist two sequences $\{v_n\}, \{x_n\}$ $n \ge 0$ satisfying approximations (4)–(7),

(14) $$v_0 \le w_0 \le v_1 \le \ldots \le w_n \le v_{n+1} \le x_{n+1} \le y_n \le \ldots \le x_1 \le y_0 \le x_0,$$

(15) $$\lim_{n \to \infty} v_n = v^*, \quad \lim_{n \to \infty} x_n = x^* \quad \text{and} \quad v^*, x^* \in D_0 \quad \text{with} \quad v^* \le x^*.$$

Moreover, if the operators $A_n = [x_n, x_n]$ are inverse nonnegative, then any solution u of the equation $F(x) = 0$ in $\langle v_0, x_0 \rangle$ belong to $\langle v^*, x^* \rangle$.

Proof. Let us define the operator

$$P_1 : \langle 0, x_0 - v_0 \rangle \to E_1, \quad P_1(x) = x - B_0(F(v_0) + A_0(x)).$$

This operator is isotone and continuous. We can have in turn

$$P_1(0) = -B_0 F(v_0) \ge 0,$$

$$P_1(x_0 - v_0) = x_0 - v_0 - B_0 F(x_0) + B_0(F(x_0) - F(v_0) - A_0(x_0 - v_0))$$
$$\le x_0 - v_0 + B_0([x_0, v_0] - [x_0, x_0])(x_0 - v_0) \quad \text{(by (9))}$$
$$\le x_0 - v_0,$$

since $[x_0, v_0] \le [x_0, x_0]$ by (12).

By Kantorovich's theorem [4], operator P_1 has a fixed point $z_1 \in \langle 0, x_0 - v_0 \rangle$: $P_1(z_1) = z_1$. Set $w_0 = v_0 + z_1$, then we have estimates

$$F(v_0) + A_0(w_0 - v_0) = 0,$$

$$F(w_0) = F(w_0) - F(v_0) - A_0(w_0 - v_0) \le 0$$

and

$$v_0 \le w_0 \le x_0.$$

We define the operator

$$P_2 : \langle 0, x_0 - w_0 \rangle \to E_1, \quad P_2(x) = x - B_0(F(x_0) + A_0(x)).$$

This operator is isotone and continuous. We can have in turn

$$P_2(0) = B_0 F(x_0) \ge 0,$$

$$P_2(x_0 - w_0) = x_0 - w_0 + B_0 F(w_0) + B_0(F(x_0) - F(w_0) - A_0(x_0 - w_0)) \le$$
$$\le x_0 - w_0 + B_0([x_0, w_0] - [x_0, x_0])(x_0 - w_0) \quad \le \text{(by (9))}$$
$$\le x_0 - w_0,$$

since $[x_0, w_0] \le [x_0, x_0]$ by (12).

By Kantorovich's theorem there exists $z_2 \in \langle 0, x_0 - w_0 \rangle$ such that $P_2(z_2) = z_2$. Set $y_0 = x_0 - z_2$, then we have the estimates

$$F(x_0) + A_0(y_0 - x_0) = 0,$$

$$F(y_0) = F(y_0) - F(x_0) - A_0(y_0 - x_0) \ge 0$$

and
$$v_0 \le w_0 \le y_0 \le x_0.$$

We now define the operator
$$P_3 : \langle 0, x_0 - v_0 \rangle \to E_1, \quad P_3(x) = x - B_0(L_0 B_0 F(v_0) + A_0(x)),$$
where $L_0 = [x_0, x_0] - [x_0, y_0]$.

This operator is isotone and continuous. We have in turn
$$P_3(0) = -B_0 L_0 B_0 F(v_0) \ge 0 \quad \text{by (9)}$$
$$P_3(x_0 - v_0) = x_0 - v_0 - B_0 L_0 B_0 F(x_0) +$$
$$+ B_0(L_0 B_0(F(x_0) - F(v_0)) - [x_0, x_0](x_0 - v_0)).$$

But, by (11) and (12), we can have
$$L_0 B_0 F(x_0) - F(v_0)) - [x_0, x_0](x_0 - v_0) =$$
$$= (L_0 B_0 [x_0, v_0] - [x_0, x_0])(x_0 - v_0) \le$$
$$\le (L_0 - [x_0, x_0])(x_0 - v_0) \le -[x_0, y_0](x_0 - v_0) \le 0.$$

Therefore, we have
$$P_3(x_0 - v_0) \le x_0 - v_0.$$

By Kantorovich's theorem there exists $z_3 \in \langle 0, x_0 - v_0 \rangle$ such that $P_3(z_3) = z_3$. Set $v_1 = w_0 + z_3$, then we have estimates
$$-L_0(w_0 - v_0) + A_0(v_0 - w_0) = 0$$
and
$$L_0(w_0 - v_0) \ge 0.$$

Furthermore, we can define the operator
$$P_4 : \langle 0, x_0 - v_0 \rangle \to E_1, \quad P_3(x) = x + B_0(L_0 B_0 F(x_0) - A_0(x)).$$

This operator is isotone and continuous. We have in turn
$$P_4(0) = B_0 L_0 B_0 F(x_0) \ge 0 \quad \text{by (9)},$$
$$P_4(x_0 - v_0) = x_0 - v_0 + B_0 L_0 B_0 F(v_0) +$$
$$+ B_0(L_0 B_0(F(x_0) - F(v_0)) - A_0(x_0 - v_0)) \le x_0 - v_0$$
(by using the same approach as for P_3). By Kantorovich's theorem there exists $z_4 \in \langle 0, x_0 - v_0 \rangle$ such that $P_4(z_4) = z_4$. Set $x_1 = y_0 - z_4$, then we have estimates
$$-L_0(y_0 - x_0) + A_0(x_1 - y_0) = 0$$

and
$$L_0(y_0 - x_0) \le 0.$$

From approximation (6) we now have
$$v_1 - w_0 = w_0 + B_0 L_0(w_0 - v_0) - w_0 = B_0 L_0(w_0 - v_0) \ge 0.$$

Hence, we obtain $w_0 \le v_0$.

Moreover, from approximation (5) we have
$$x_1 - y_0 = y_0 + B_0 L_0(y_0 - x_0) - y_0 = B_0 L_0(y_0 - x_0) \le 0.$$

That is, we get $x_1 \le y_0$.

Furthermore, we can obtain in turn
$$v_1 - x_1 = w_0 + B_0 L_0(w_0 - v_0) - (y_0 + B_0 L_0(y_0 - x_0)) =$$
$$= w_0 - y_0 + B_0 L_0(w_0 - v_0 + x_0 - y_0) =$$
$$= v_0 - B_0 L_0 F(v_0) - (x_0 - B_0 F(x_0)) +$$
$$+ B_0 L_0(v_0 - B_0 F(v_0 - B_0 F(v_0)) - B_0 L_0(v_0) +$$
$$+ B_0 L_0(x_0) - B_0 L_0(x_0 - B_0 F(x_0)) =$$
$$= v_0 - x_0 - B_0(F(v_0) - F(x_0)) - B_0 L_0 B_0(F(v_0) - F(x_0)) =$$
$$= (I - B_0[v_0, x_0] - B_0 L_0 B_0 [v_0, x_0])(v_0 - x_0).$$

But using hypotheses (12) and (13) we have
$$B_0 L_0 B_0 [v_0, x_0] + B_0 [v_0, x_0] \le B_0 L_0 B_0 A_0 + B_0 [v_0, x_0] \le$$
$$\le B_0 L_0 + B_0 [v_0, x_0] \le B_0(L_0 + [v_0, x_0]) \le$$
$$\le B_0 [y_0, q] \le B_0 A_0 \le I.$$

We now obtain $v_1 \le x_1$.

From all the above we have
$$v_0 \le w_0 \le v_1 \le x_1 \le y_0 \le x_0.$$

By hypothesis (12), it follows that the operator A_n has a continuous nonnegative left subinverse B_n for all $n \ge 0$. Proceeding by induction, we can show that there exist two sequences $\{v_n\}, \{x_n\}$ $(n \ge 0)$ satisfying (4) – (7) and (14) is a regular space E_1 and as such they converge to some $v^*, x^* \in D_0$. That is, we have
$$\lim_{n \to \infty} v_n = v^* \le x^* = \lim_{n \to \infty} x_n.$$

If $v_0 \le u \le x_0$ and $F(u)=0$, then we can obtain

$$A_0(y_0-u)=A_0(x_0-B_0F(x_0))-A_0u=$$
$$=A_0(x_0-u)-A_0B_0(F(x_0)-F(u))=$$
$$=A_0(I-B_0[x_0,u])(x_0-u)\ge 0,\quad \text{since } B_0[x_0,u]\le B_0A_0\le I.$$

Similarly, we show $A_0(w_0-u)\le 0$.

If the operator A_0 is inverse nonnegative, then it follows from the above $w_0\le u\le y_0$. Proceeding by induction, we deduce that $w_n\le u\le y_n$, from which it follows that $w_n\le v_n\le w_{n+1}\le u\le y_{n+1}\le x_n\le y_n$, for all $n\ge 0$. That is, we have $v_n\le u\le x_n$ for all $n\ge 0$. Hence, we get $v^*\le u\le x^*$.

That completes the proof of the theorem.

In what follows, we shall give some natural conditions under which the points v^* and x^* are solutions of equation $F(x)=0$.

THEOREM 2. *Under hypotheses of Theorem 1 suppose F is continuous at* v^* *and* x^*. *If one of the folowing conditions is satisfied*

(a) $x^*=y^*$,

(b) E_1 *is normal and there exists an operator* $Q:E_1\to E_2$, ($Q(0)=0$) *which has an isotone inverse continuous at the origin and such that* $A_n\le T$ *for sufficiently large n,*

(c) E_2 *is normal and there exists an operator* $R:E_1\to E_2(R(0)=0)$ *continuous at the origin and such that* $A_n\le R$ *for sufficiently larg n,*

(d) *operators* A_n *are equicontinuous for all* $n\ge 0$,

and

(e) E_2 *is normal and* $[u,v]\le[x,y]$ *if* $u\le x$ *and* $v\le y$,

then we have

$$F(v^*)=F(x^*)=0.$$

Proof. (a) Using the continuity of F and $F(v_n)\le 0\le F(x_n)$, we get $F(v^*)\le 0\le F(v^*)$. That is, we obtain $F(x^*)=F(v^*)=0$.

(b) By (4) and (6)

$$0\ge F(v_n)=A_n(v_n-w_n)\ge Q(v_n-w_n)$$

$$0\le F(x_n)=A_n(x_n-y_n)\ge Q(x_n-y_n).$$

Hence, we get

$$0\ge Q^{-1}F(v_n)\ge v_n-w_n,\quad 0\le Q^{-1}F(x_n)\le x_n-y_n.$$

Since E_1 is normal and $\lim_{n\to\infty}(v_n-w_n)=\lim_{n\to\infty}(x_n-y_n)=0$, we have $\lim_{n\to\infty}Q^{-1}F(v_n)=$ $=\lim_{n\to\infty}Q^{-1}F(x_n)=0$. Hence, by continuity, we get $F(v^*)=F(x^*)=0$.

(c) As above, we get

$$0\ge F(v_n)\ge R(v_n-w_n),\quad 0\le F(x_n)\le R(x_n-y_n).$$

Using the normality of E_2 and the continuity of F and R, we get $F(v^*)=F(x^*)=0$.

(d) From the equicontinuity of the operator A_n we have $\lim_{n\to\infty}A_n(v_n-w_n)=$ $=\lim_{n\to\infty}A_n(x_n-y_n)=0$. Hence, by (4) and (6)

$$F(v^*)=F(x^*)=0.$$

(e) Using hypotheses (10) – (14), we get in turn

$$0\le F(y_n)=F(y_n)-F(x_n)-A_n(y_n-x_n)=$$
$$=(A_n-[y_n,x_n])(x_n-y_n)\le([x_0,x_0]-[x^*,x^*])(x_n-y_n).$$

Since E_2 is normal and $\lim_{n\to\infty}(x_n-y_n)=0$, we get $\lim_{n\to\infty}F(x_n)=0$. Moreover, from hypothesis (12)

$$[x^*,x^*](x_n-x^*)\le[x^*,x_n](x_n-x^*)=$$
$$=F(x_n)-F(x^*)\le[x_0,x_0](x_n-x^*)$$

and by the normality of $E_2, F(x^*)=\lim_{n\to\infty}F(x_n)$. Hence, we get $F(x^*)=0$. The result $F(v^*)=0$ can be obtained similarly.

The proof of the theorem is now complete.

As in Theorems 1 and 2, we can prove the following result (see also [7, Theorem 6.2]:

THEOREM 3. *Assume that hypotheses of Theorem 1 are true. Then the approximations*

$$y_n=x_n-B_nF(x_n),$$
$$x_{n+1}=y_n+B_nL_n(y_n-x_n),\quad L_n=[x_n,x_n]-[x_n,y_n]$$
$$w_n=v_n-B_nF(v_n)$$

and

$$v_{n+1} = w_n + B_n L_n (w_n - v_n)$$

where the operators B_n *are nonnegative subinverses of* A_n, *generate two sequences* $\{v_n\}$ *and* $\{x_n\}$ $(n \ge 0)$ *satisfying approximations* (4) – (7) *and* (14). *Moreover, for any solution* $u \in \langle v_0, x_0 \rangle$ *of the equation* $F(x) = 0$ *we have*

$$u \in \langle v_n, x_n \rangle \quad (n \ge 0).$$

Furthermore, assume that the following are true:

(a) E_2 *is a POTL-space and* E_1 *is a normal POTL-space;*

(b) $\lim_{n \to \infty} x_n = x^*$ *and* $\lim_{n \to \infty} v_n = v^*$;

(c) F *is continuous at* v^* *and* x^*;

and

(d) *there exists a continuous nonsingular nonnegative operator* T *such that* $B_n \ge T$ *for sufficiently large n. Then*

$$F(v^*) = F(x^*) = 0.$$

Remarks. (a) Our conditions coincide with (44) and (50) in [7, p. 98]. In case $E_1 = E_2 = \mathbb{R}$, our conditions (12) and (13) are satisfied if and only if F is differentiable on D_0, and F, F' are convex on D_0.

(b) It follows from all the above that our method uses the same or simpler conditions than those used in all previous results ([4] – [9]) but the order of convergence is faster [3].

(c) Similar results can immediately follow if the divided difference $[x_0, x_0]$ is replaced by $[x_0, x_0] v_0 \le z_0 \le x_0$ in (10), $[x_n, x_n]$ is replaced by $[x_n, y_{n-1}]$ $(n \ge 1)$ in (4) – (7).

REFERENCES

1. I. K. Argyros, and F. Szidarovszky, *On the monotone convergence of general Newton-like methods*, Bull. Austral. Math. Soc., **45**, (1992), 489–502.
2. I. K. Argyros, and F. Szidarovszky, *The Theory and Applications of Iteration Methods*, C.R.C. Press, Inc. Boca Raton, Florida, U.S.A. (1993).
3. I. K. Argyros, *On the convergence of Chebysheff-Halley-type method under Newton-Kantorovich hypotheses*, Appl. Math. Letter, **6**, 5, (1993), 71–74.
4. L .V. Kantorovich, *The method of successive approximation for functional equations*, Acta Math., **71** (1939), 63–97.
5. M. A. Mertvecova, *An analog of the process of tangent hyperbolas for general functional equations* (in Russian), Dokl. Akad. Nauk SSSR, **88** (1953), 611–614.

6. M. T. Necepurenko, *On Chebysheff's method for functional equations* (in Russian), Usephi Mat. Nauk., **9** (1954), 163–170.
7. F. A. Potra, *On an iterative algorithm of order 1.839 ... for solving nonlinear operator equations*, Numer. Funct. Anal. Optimiz., **7** (*1*), (1984–1985), 75–106.
8. S. Ul'm, *Iteration methods with divided differences of the second order* (in Russian), Dokl. Akad. Nauk SSSR, **158** (1964), 55–58. Soviet Math. Dokl. *5*, 1187–1190.
9. J. A. Vandergraft, *Newton's method for convex operators in partially ordered spaces*, SIAM J. Numer. Anal. **4** (1967), 406 – 432.

Received August 10, 1996

Department of Mathematics,
Cameron University, Lawton, OK 73505,
U.S.A.