REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome XXVII, N° 1, 1998, pp. 33–45

ON SOME INTERPOLATORY ITERATIVE METHODS FOR THE SECOND DEGREE POLYNOMIAL OPERATORS (I)

EMIL CĂTINAŞ, ION PĂVĂLOIU

1. INTRODUCTION

The most well-known interpolatory iterative methods have been studied by several authors ([3], [4], [8], [13], [14], [15], [16] and [19]) also in the case of operator equations. These methods have the advantage of a higher efficiency when compared with the methods that use the Fréchet derivatives of the corresponding operators, and, moreover, they may be applied even when the Fréchet derivative vanishes at some points in the neighborhood of the solution ([2] and [16]). On the other hand, the construction of the finite difference operators may be difficult for a large class of general operators.

In this note we shall consider second degree polynomial operators; for these operators the divided differences at any four points are the null trilinear operators. These equations belong to an important class which has many applications in practice [2].

In Sections 2 and 3 we construct the divided differences of some particular operators, and the Lagrange interpolatory polynomial in the Newton form. In Section 4 we study the convergence of two interpolatory iterative methods, namely the chord method and the Steffensen method. We have considered that this study may present some interest because of the conditions for the convergence, which are simpler than in the general case.

2. DIVIDED DIFFERENCES

Let **X** and **Y** be two normed linear spaces and $F:\mathrm{X}\to\mathrm{Y}$ a mapping. Denote by $\mathscr{L}(\mathbf{X},\mathbf{Y})$ the set of linear continuous operators from **X** into **Y** and by $\mathscr{L}(\mathbf{X}^i,\mathbf{Y})$ the set of i-linear continuous operators from $\mathbf{X}^i$ into **Y**.

1991 AMS Subject Classification: 65J20, 65H17.

DEFINITION 1 [14]. *Given the distinct points* $x_1, x_2 \in \mathbf{X}$, *the mapping* $[x_1, x_2; F] \in \mathscr{L}(\mathbf{X}, \mathbf{Y})$ *is called the first order divided difference of F on the points* x_1, x_2 *if:*

a) $[x_1, x_2; F](x_2 - x_1) = F(x_2) - F(x_1)$;

b) *if F is Fréchet differentiable at* x_s, *then* $[x_s, x_s; F] = F'(x_s)$.

The higher order divided differences are constructed recursively.

DEFINITION 2. [14]. *Given the distinct points* $x_1, \dots, x_{i+1} \in \mathbf{X}, i \geq 2$, *the mapping* $[x_1, \dots, x_{i+1}; F] \in \mathscr{L}(\mathbf{X}^i, \mathbf{Y})$ *is called the i-th order divided difference of F on the points* $x_1, \dots, x_{i+1}$ *if:*

a) $[x_1, \dots, x_{i+1}; F](x_{i+1} - x_1) = [x_2, \dots, x_{i+1}; F] - [x_1, \dots, x_i; F]$;

b) *if there exists the i-th order Fréchet derivative of F on* $x_s \in \mathbf{X}$, *then*

$$[x_s, \dots, x_s; F] = \frac{1}{i!} F^{(i)}(x_s).$$

DEFINITION 3. [14]. *The divided difference* $[x_1, \dots, x_{i+1}; F]$ *is called symmetric with respect to* $x_1, \dots, x_{i+1}$ *if the equalities*

$$[x_1, \dots, x_{i+1}; F] = [x_{k_1}, \dots, x_{k_{i+1}}; F] \tag{2.1}$$

hold for any permutation $(k_1, \dots, k_{i+1})$ *of* $(1, 2, \dots, i+1)$.

Remark. When $\mathbf{X} = \mathbf{R}$ then it is well known that the equalities (2.1) hold. However, these equalities do not generally hold for any normed space **X**.

Example 1. Let $\mathbf{X} = \mathbf{R}^2$, $\mathbf{Y} = \mathbf{R}$ and $F : \mathbf{R}^2 \to \mathbf{R}$. Denote $x_i = (u_i, v_i) \in \mathbf{R}^2$, $i = 1, 2$, $u_1 \neq u_2$ and $v_1 \neq v_2$. Any of the following two expressions defines a first order divided difference of F on x_1 and x_2:

$$[x_2, x_1; F] = \left(\frac{F(u_1, v_1) - F(u_2, v_1)}{u_1 - u_2} \quad \frac{F(u_2, v_1) - F(u_2, v_2)}{v_1 - v_2} \right)$$

$$[x_1, x_2; F] = \left(\frac{F(u_2, v_2) - F(u_1, v_2)}{u_2 - u_1}, \quad \frac{F(u_1, v_2) - F(u_1, v_1)}{v_2 - v_1} \right).$$

It is obvious that, in general $[x_1, x_2; F] \neq [x_2\, x_1; F]$, which means that these divided differences are not symmetric with respect to the points x_1, x_2.

For symmetric divided differences we may consider in this case the expression

$$[x_1, x_2; F] = \frac{1}{2}\left(\left[\frac{F(u_1, v_1) - F(u_2, v_1)}{u_1 - u_2} + \frac{F(u_2, v_2) - F(u_1, v_2)}{u_2 - u_1} \right], \right.$$
$$\left. \left[\frac{F(u_2, v_1) - F(u_2, v_2)}{v_1 - v_2} + \frac{F(u_1, v_2) - F(u_1, v_1)}{v_2 - v_1} \right] \right).$$

Example 2. Let **V** be a Banach space over the field **K** (**K** = **R** or **C**) and consider $A \in \mathscr{L}(\mathbf{V})$ a linear continuous operator on **V**. The scalar $\lambda \in \mathbf{K}$ is an eigenvalue of A iff the equation

$$Av - \lambda v = \theta$$

has at least a solution $v \neq \theta$. In this case, v is an eigenvector of A corresponding to λ.

For determining an eigenpair (v, λ), consider a linear continuous mapping $G : \mathbf{V} \to \mathbf{K}$ and the Banach space $\mathbf{X} = \mathbf{V} \times \mathbf{K}$ with the norm $\|x\| = \max\{\|v\|, |\lambda|\}$, $x = \begin{pmatrix} v \\ \lambda \end{pmatrix}$, $v \in \mathbf{V}, \lambda \in \mathbf{K}$. Let $F : \mathbf{X} \to \mathbf{X}$ be defined

$$F(x) = \begin{pmatrix} Av - \lambda v \\ Gv - 1 \end{pmatrix}. \tag{2.2}$$

Denoting by $\theta_1 = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$ the null element of **X**, then the eigenpairs are the solutions of the equation

$$F(x) = \theta_1.$$

We shall construct for F the divided differences, and we shall show that for $i \geq 3$ the i-th order divided differences are the i-linear null operators.

Remark. When $\mathbf{V} = \mathbf{K}^n$, it is usually considered the quadratic function $G(v) = \frac{1}{2} v^t v$, instead of a linear one. We have shown in [3] and [4] that for the choice $G(v) = \frac{1}{2n} v^t v$ the norm of the second derivative has the constant value 2, smaller when compared to the constant value n, which corresponds to the second quadratic function.

The divided differences of the corresponding operators F may be easily constructed.

Let $x_i = \begin{pmatrix} v_i \\ \lambda_i \end{pmatrix} \in \mathbf{X}, i=\overline{1,3}$ and $k_i = \begin{pmatrix} h_i \\ \alpha_i \end{pmatrix} \in \mathbf{X}, i=1,2.$

For determining the first order divided differences of F we have that

$$F(x_2)-F(x_1) = \begin{pmatrix} A(v_2-v_1) - \frac{1}{2}[(\lambda_1+\lambda_2)(v_2-v_1)+(\lambda_2-\lambda_1)(v_2+v_1)] \\ G(v_2-v_1) \end{pmatrix}$$

whence it follows that

$$[x_1,x_2;F]\,k_1 = \begin{pmatrix} Ah_1 - \frac{1}{2}((\lambda_1+\lambda_2)h_1+\alpha_1(v_2+v_1)) \\ Gh_1 \end{pmatrix}.$$

Using the above formula, we get for the second order divided differences of F that

$$[x_1,x_2,x_3;F]\,k_1k_2 = \begin{pmatrix} -\frac{1}{2}(\alpha_2 h_1+\alpha_1 h_2) \\ 0 \end{pmatrix}.$$

Since the above divided difference does not depend on $x_i, i=\overline{1,3}$, it follows that the higher order divided differences are the null multilinear operators.

Consider now the matrix $A \in \mathcal{M}_n(\mathbf{K})$. For $\mathbf{V}=\mathbf{K}^n$ and $\mathbf{X}=\mathbf{K}^{n+1}$, the operator (2.2) is given by the relations

(2.3) $$F_i(x) = F_i(x^1,\ldots,x^{n+1}),\ i=\overline{1,n+1}$$

where

$$F_i(x) = a_{i1}x^1+\ldots+a_{i,i-1}x^{i-1}+(a_{ii}-x^{n+1})x^i+a_{i,i+1}x^{i+1}+\ldots+a_{in}x^n,\ i=\overline{1,n}$$

and for $G(v)-1$ we may take

$$F_{n+1}(x) = x^{i_0}-1,$$

where $i_0 \in \{1,\ldots,n\}$ is fixed. We have denoted $x=(x^1,\ldots,x^{n+1})$, x^{n+1} being an unknown eigenvalue of A, and $v=(x^1,\ldots,x^n)$ a corresponding eigenvector.

The matrices associated to the first order divided differences of F on the points $x_1, x_2 \in \mathbf{K}^{n+1}$ are

$$[x_1,x_2;F] = \begin{pmatrix} b_{11} & a_{12} & \ldots & a_{1i_0} & \ldots & a_{1n} & a_{1,n+1} \\ a_{21} & b_{22} & \ldots & a_{2i_0} & \ldots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{ni_0} & \ldots & b_{nn} & a_{n,n+1} \\ 0 & 0 & \ldots & 1 & \ldots & 0 & 0 \end{pmatrix}$$

with $b_{ii} = a_{ii} - \frac{1}{2}(x_2^{n+1}+x_1^{n+1}), i=\overline{1,n}$ and $a_{i,n+1} = -\frac{1}{2}(x_2^i+x_1^i), i=\overline{1,n}$.

Consider now $x_3 \in \mathbf{K}^{n+1}$ and $h_1=(h_1^1,\ldots,h_1^{n+1})$, $h_2=(h_2^1,\ldots,h_2^{n+1}) \in \mathbf{K}^{n+1}$.

The second order divided differences of F on x_1, x_2, x_3 are

$$[x_1,x_2,x_3;F]\,h_1h_2 = \begin{pmatrix} -\frac{1}{2}h_2^{n+1} & 0 & \ldots & 0 & -\frac{1}{2}h_2^1 \\ 0 & -\frac{1}{2}h_2^{n+1} & \ldots & 0 & -\frac{1}{2}h_2^2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & -\frac{1}{2}h_2^{n+1} & -\frac{1}{2}h_2^n \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \begin{pmatrix} h_1^1 \\ h_1^2 \\ \vdots \\ h_1^n \\ h_1^{n+1} \end{pmatrix}$$

for all $h_1, h_2 \in \mathbf{K}^{n+1}$.

It can be easily seen that the above finite differences are symmetric with respect to the points considered.

Example 3. Let $T:\mathbf{X} \to \mathbf{X}$ be given by

$$T(x) = y + B_1(x) + B_2(x),$$

with $y \in \mathbf{X}$ fixed, $B_1 \in \mathcal{L}(\mathbf{X})$ and B_2 the restriction to the diagonal of $\mathbf{X} \times \mathbf{X}$ of a bilinear symmetric operator $\overline{B}_2 \in \mathcal{L}(\mathbf{X}^2, \mathbf{X})$, i.e., $B_2(x) = \overline{B}_2(x,x)$.

Let $x_i \in \mathbf{X}, i=\overline{1,3}$ be three points. The first order divided differences of T on x_1 and x_2 are

$$[x_1,x_2;T]\,h_1 = B_1(h_1) + \overline{B}_2(x_1+x_2, h_1), \quad h_1 \in \mathbf{X},$$

while the second order ones are given by

$$[x_1,x_2,x_3;T]\,h_1h_2 = \overline{B}_2(h_1,h_2), \quad h_1, h_2 \in \mathbf{X}.$$

It is clear that, in this case too, the higher order divided differences are the null multilinear operators.

Example 4. Let $\mathbf{X}=\mathbf{Y}=C[0,1]$ be the Banach space of continuous functions on $[0,1]$, equipped with the max norm. We consider the mapping $F: C[0,1] \to C[0,1]$ given by

$$F(x)(s) = \int_0^1 K(s,t,x(t))\,dt,$$

where $K:[0,1]\times[0,1]\times\mathbf{R} \to \mathbf{R}$ is a continuous function.

The first order divided differences of F on the points $x_1, x_2 \in C[0,1]$, when $x_1(t) \neq x_2(t)$, $t \in [0,1]$ are given by

$$[x_1,x_2;F]\,h = \int_0^1 \frac{K(s,t,x_2(t)) - K(s,t,x_1(t))}{x_2(t)-x_1(t)}\,h(t)\,dt, \quad h \in C[0,1].$$

Indeed, it can be seen that

$$[x_1, x_2; F](x_2 - x_1) = \int_0^1 \frac{K(s,t,x_2(t)) - K(s,t,x_1(t))}{x_2(t) - x_1(t)}(x_2(t) - x_1(t))\, dt =$$
$$= F(x_2) - F(x_1),$$

and so relation a) from Definition 2.1 is satisfied.

If K admits a partial derivative with respect to the third argument on $\mathbf{R}$, then relation b) from Definition 2.1 is also verified for all $(s,t,x) \in [0,1]\times[0,1]\times\mathbf{R}$.

It can be easily seen that these divided differences are symmetric with respect to x_1 and x_2.

For the higher order divided differences we consider $x_i \in C[0,1]$, $i = \overline{1,n+1}$ with $x_i(t) \neq x_j(t)$ for $i \neq j$ and $t \in [0,1]$.

Denote

$$[x_i, x_{i+1}; K] = \frac{K(s,t,x_{i+1}(t)) - K(s,t,x_i(t))}{x_{i+1}(t) - x_i(t)}, \quad i = 1,\ldots,n,$$

$$[x_i, x_{i+1}, x_{i+2}; K] = \frac{[x_{i+1}, x_{i+2}; K] - [x_i, x_{i+1}; K]}{x_{i+2}(t) - x_i(t)}, \quad i = 1,\ldots,n-1,$$

$$[x_i, x_{i+1},\ldots, x_{i+s}; K] = \frac{[x_{i+1},\ldots, x_{i+s}; K] - [x_i,\ldots, x_{i+s-1}; K]}{x_{i+s}(t) - x_i(t)}, \quad i = 1,\ldots,n-s+1.$$

With these notations, we easily obtain that

$$[x_i, x_{i+1}, x_{i+2}; F]h_1 h_2 = \int_0^1 [x_i, x_{i+1}, x_{i+2}; K]h_1(t)\, h_2(t)\, dt, \quad i = \overline{1,n-1}$$

and, in general,

$$[x_i,\ldots, x_{i+s}; F]h_1\ldots h_s = \int_0^1 [x_i,\ldots, x_{i+s}; K]h_1(t)\ldots h_s(t)\, dt, \quad i = \overline{1,n-s+1}.$$

3. INTERPOLATION

Let $F: \mathbf{X} \to \mathbf{Y}$ be a mapping and $x_i \in \mathbf{X}, i = \overline{1,n+1}$, some distinct points. Consider the polynomial operator $L_n \in \mathbf{X} \to \mathbf{Y}$ given by

(3.1)
$$L_n(x) = F(x_1) + [x_1, x_2; F](x - x_1) + \\ + \ldots + [x_1, x_2, \ldots, x_{n+1}; F](x - x_n)\ldots(x - x_1).$$

The following result holds.

THEOREM 3.1. [14]. *If the mapping F admits symmetric divided differences from the first order to the $n+1$-th order with respect to the points $x_i, x \in \mathbf{X}$, $i = \overline{1,n+1}$, then the following relations hold:*

(3.2)
$$L_n(x_i) = F(x_i), \quad i = \overline{1,n+1}$$

(3.3)
$$F(x) - L_n(x) = [x, x_1, \ldots, x_{n+1}; F](x - x_{n+1})\ldots(x - x_1).$$

Let $D \subseteq \mathbf{X}$ be an open subset and suppose that the restriction $G = F|_D$ is a homeomorphism between D and $F(D)$. The following result can be easily obtained.

LEMMA 3.1. [14]. *If the mapping $[x_1, x_2; G] \in \mathscr{L}(\mathbf{X},\mathbf{Y}), x_1, x_2 \in D$ is invertible, then*

$$[y_1, y_2; G^{-1}] = [x_1, x_2; G]^{-1},$$

where G^{-1} is the inverse of G and $y_i = G(x_i), i = 1,2$.

4. ITERATIVE METHODS FOR SECOND DEGREE POLYNOMIAL OPERATOR EQUATIONS

In the following we shall consider second degree polynomial operator equations

(4.1)
$$F(x) = 0,$$

with $F: \mathbf{X} \to \mathbf{X}$ which satisfies then

(4.2)
$$[x, y, z, t; F] = \theta_3, \quad \forall x, y, z, t \in \mathbf{X},$$

θ_3 being the trilinear null operator.

It follows that F also satisfies

(4.3)
$$F(x) = F(x_1) + [x_1, x_2; F](x - x_1) + [x_1, x_2, x; F](x - x_2)(x - x_1),$$

where $[x_1, x_2, x; F]$ does not depend on x_1, x_2 and x (see also Examples 2.2 and 2.3).

We are interested in the convergence of the chord method and Steffensen method for this equation.

The chord method is given by the iteration

(4.4)
$$x_{k+1} = x_{k-1} - [x_{k-1}, x_k; F]^{-1} F(x_{k-1}), \quad k = 1,2,\ldots, x_0, x_1 \in \mathbf{X}$$

or, equivalently,

(4.5) $$x_{k+1}=x_k-[x_{k-1},x_k;F]^{-1}F(x_k),\ k=1,2,\ldots,x_0,x_1\in\mathbf{X}.$$

For the Steffensen method, it is considered an auxiliary function $g:\mathbf{X}\to\mathbf{X}$ and the equation

(4.6) $$x-g(x)=0,$$

equivalent to (4.1). The sequence $(x_k)_{k\geq 0}$ is then constructed by the iteration

(4.7) $$x_{k+1}=x_k-[x_k,g(x_k);F]^{-1}F(x_k),\ \ k=0,1,\ldots,x_0\in\mathbf{X}$$

or

(4.8) $$x_{k+1}=g(x_k)-[x_k,g(x_k);F]^{-1}F(g(x_k)),\ \ k=0,1,\ldots,x_0\in\mathbf{X}.$$

It can be easily verified that the two relations above define the same sequence.

Because of the special form of equation (4.1), we shall see that the conditions for the convergence of the sequences generated by (4.4) – (4.5) and (4.7) – (4.8) are much more simplified as compared to the general case of an arbitrary equation (4.1).

The convergence of the chord method. Let $\alpha=\|[x,y,z;F]\|$ and $x_0\in\mathbf{X}$. Denote $B(x_0,r)=\{x\in\mathbf{X}|\ \|x-x_0\|\leq r\},r>0$. Consider $x_1\in B(x_0,r)$ and $d_0=\|x_1-x_0\|$. The following result holds.

THEOREM 4.1. *If the mapping F and the elements $x_0,x_1\in\mathbf{X}$, and $\alpha,r,b_0,d_0\in\mathbf{R}_+$ satisfy*

i) *there exists* $[x_0,x_1;F]^{-1}$

ii) $b_0\alpha(2r+d_0)=q<1$, *with* $b_0=\|[x_0,x_1;F]^{-1}\|$

iii) $\alpha b^2\|F(x_0)\|=\varepsilon_0<1,\quad \alpha b^2\|F(x_1)\|\leq\varepsilon_0^l$, *where* $b=\dfrac{b_0}{1-q}$ *and* $l=\dfrac{1+\sqrt{5}}{2}$

iv) $\dfrac{\varepsilon_0^l}{\alpha b(1-\varepsilon_0)}+d_0\leq r,$

then the sequence $(x_k)_{k\geq 0}$ generated by (4.4) *is well defined and its elements belong to $B(x_0,r)$. Moreover, $(x_k)_{k\geq 0}$ is convergent, and denoting $x^*=\lim\limits_{k\to\infty}x_k$, then $F(x^*)=0$. The following estimation also holds:*

$$\|x^*-x_k\|\leq\frac{\varepsilon_0^{l^k}}{\alpha b(1-\varepsilon_0^l)}.$$

Proof. We shall first prove that, if $x,y\in B(x_0,r)$, then there exists $[x,y;F]^{-1}$ and

(4.9) $$\|[x,y;F]^{-1}\|\leq b.$$

Let $T=[x_0,x_1;F]^{-1}([x_0,x_1;F]-[x,y;F])$.

It can be easily seen that

$$\|T\|\leq b_0\alpha(2r+d_0),$$

whence by ii) it follows the existence of $(I-T)^{-1}$ and the inequality (4.9).

Now we show that $x_2\in B(x_0,r)$. From i) it follows that x_2 is well defined, and from (4.5), for $k=1$ we get

$$\|x_2-x_1\|\leq b\|F(x_1)\|=\frac{\varepsilon_0^l}{\alpha b}.$$

Using this inequality and the hypothesis iv) one obtains

$$\|x_2-x_0\|\leq\|x_2-x_1\|+\|x_1-x_0\|\leq\frac{\varepsilon_0^l}{\alpha b(1-\varepsilon_0)}+d_0\leq r,$$

i.e., $x_2\in B(x_0,r)$.

Now, using (4.3) with $x=x_2$ and taking into account (4.4) – (4.5), it follows that

$$\|F(x_2)\|\leq\alpha b^2\|F(x_1)\|\cdot\|F(x_0)\|,$$

whence, by iii),

$$\|F(x_2)\|\leq\frac{1}{\alpha b^2}\varepsilon_0^{1+l}=\frac{1}{\alpha b^2}\varepsilon_0^{l^2}.$$

Let $k\in\mathbf{N}$ and suppose that $x_{k-1},x_k\in B(x_0,r)$, and

$$\|F(x_{k-1})\|\leq\frac{1}{\alpha b^2}\varepsilon_0^{l^{k-1}}$$

$$\|F(x_k)\|\leq\frac{1}{\alpha b^2}\varepsilon_0^{l^k}.$$

It easily follows that x_{k+1} is well defined, that

$$\|x_{k+1}-x_k\|\leq b\|F(x_k)\|\leq\frac{1}{\alpha b^2}\varepsilon_0^{l^k}$$

and that

$$\|x_{k+1}-x_0\|\le\sum_{i=0}^{k}\|x_{i+1}-x_i\|=\|x_1-x_0\|+\sum_{i=1}^{k}\|x_{i+1}-x_i\|\le$$

$$\le d_0+\frac{1}{\alpha b}\sum_{i=1}^{k}\varepsilon_0^{l^i}\le d_0+\frac{\varepsilon_0^{l}}{\alpha b(1-\varepsilon_0)}\le r,$$

i.e., $x_{k+1}\in B(x_0,r)$.

Using (4.3) we get

$$\|F(x_{k+1})\|\le\alpha b^2\|F(x_k)\|\cdot\|F(x_{k-1})\|\le\frac{1}{\alpha b^2}\varepsilon_0^{(l+1)l^{k-1}}=\frac{1}{\alpha b^2}\varepsilon_0^{l^{k+1}}.$$

It can be easily shown that $(x_k)_{k\ge0}$ is a Cauchy sequence, hence it converges and satisfies the stated estimation.

The convergence of the Steffensen method. We shall consider that the mapping g from (4.6) is given by

(4.10) $$g(x)=x-\lambda F(x),$$

with $\lambda\in\mathbf{K}$ a fixed scalar.

In this assumption, the following relations are obvious:

(4.11) $$F(g(x))-F(x)=-\lambda[x,g(x);F]\,F(x)$$

(4.12) $$[x,y;g]=I-\lambda[x,y;F],$$

I being the identity mapping on $\mathbf{X}$.

From (4.11) it follows that

(4.13) $$\|F(g(x))\|\le\|I-\lambda[x,g(x);F]\|\cdot\|F(x)\|.$$

Denote again by α the constant expression $\|[x,y,z;F]\|$, and consider $x_0\in\mathbf{X}$.

Assume that

(4.14) $$\|I-\lambda[x,y;F]\|\le\gamma,\quad\text{for all } x,y\in B(x_0,r),$$

and some $\gamma>0$. The following theorem holds.

THEOREM 4.2. *If $x_0\in\mathbf{X}$, the mapping F and the positive numbers α,β,γ,r are such that*

i) *$g(x_0)\in B(x_0,r)$ and there exists $[x_0,g(x_0);F]^{-1}$,*

ii) *$p=\beta\alpha(\gamma+1)r<1$, with $\beta=\|[x_0,g(x_0);F]^{-1}\|$,*

iii) *$\delta_0=\alpha\gamma\delta^2\|F(x_0)\|<1$, with $\delta=\dfrac{\beta}{1-p}$,*

iv) $\max\left\{\dfrac{\delta_0}{\alpha\gamma\delta(1-\delta_0)},\dfrac{\delta_0}{\alpha\delta}\left(\dfrac{1}{1-\delta_0}+\dfrac{|\lambda|}{\gamma\delta}\right)\right\}\le r,$

then the following relations hold:

j) *the sequence $(x_k)_{k\ge0}$ given by (4.7) is well defined and converges;*

jj) *denoting $x^*=\lim_{k\to\infty}x_k$, then $F(x^*)=0$;*

jjj) $\|x^*-x_k\|\le\dfrac{\delta_0^{2^k}}{\alpha\delta\gamma(1-\delta_0)},\ k=0,1,\ldots$

Proof. For $x,g(x)\in B(x_0,r)$, using that

$$\|[x_0,g(x_0);F]^{-1}([x_0,g(x_0);F]-[x,g(x);F])\|\le\beta\alpha(\gamma+1)r=p<1,$$

it follows that there exists $[x,g(x);F]^{-1}$ and

(4.15) $$\|[x,g(x);F]^{-1}\|\le\frac{\beta}{1-p}=\delta.$$

We shall use the identity

(4.16) $$F(y)=F(x)+[x,g(x);F](y-x)+$$
$$+[y,x,g(x);F](y-x)(y-g(x)),\quad\forall x,y\in\mathbf{X}.$$

From (4.7), for $k=0$ we get

$$\|x_1-x_0\|\le\beta\|F(x_0)\|\le\delta\|F(x_0)\|\le\frac{\alpha\gamma\delta^2}{\alpha\gamma\delta}\|F(x_0)\|\le\frac{\delta_0}{\alpha\gamma\delta(1-\delta_0)}\le r,$$

i.e., $x_1\in B(x_0,r)$.

Further,

$$\|g(x_1)-g(x_0)\|\le\|g(x_1)-g(x_0)\|+$$
$$+\|g(x_0)-x_0\|\le\gamma\|x_1-x_0\|+|\lambda|\|F(x_0)\|\le$$
$$\le\frac{\delta_0}{\alpha\delta(1-\delta_0)}+\frac{|\lambda|}{\gamma\alpha\delta^2}\delta_0\le\frac{\delta_0}{\alpha\delta}\left(\frac{1}{1-\delta_0}+\frac{|\lambda|}{\gamma\delta}\right)\le r,$$

which shows that $g(x_1)\in B(x_0,r)$. Taking into account (4.6), it results that

$$F(x_1)=F(x_0)+[x_0,g(x_0);F](x_1-x_0)+$$
$$+[x_1,x_0,g(x_0);F](x_1-x_0)(x_1-g(x_0))$$

and by (4.7), (4.8) and (4.11) we get

$$\|F(x_1)\| \le \alpha\delta^2\gamma\, \|F(x_0)\|^2,$$

whence for $\delta_1 = \alpha\delta^2\gamma\, \|F(x_1)\|$ we obtain

$$\delta_1 \le \delta_0^2.$$

Suppose now that $x_k, g(x_k) \in B(x_0, r)$ and that

$$\|F(x^i)\| \le \frac{\delta_0^{2^i}}{\alpha\delta^2\gamma}, \quad i = \overline{1,k}.$$

Then, obviously, x_{k+1} is well defined and

$$\|x_{k+1} - x_k\| \le \delta\, \|F(x_k)\|,$$

and

$$\|x_{k+1} - g(x_k)\| \le \delta\gamma\, \|F(x_k)\|.$$

For proving that $x_{k+1}, g(x_{k+1}) \in B(x_0, r)$, first we have for x_{k+1} that

$$\|x_{k+1} - x_0\| \le \sum_{i=0}^{k} \|x_{i+1} - x_i\| \le \sum_{i=0}^{k} \delta\, \|F(x_i)\| \le$$

$$\le \sum_{i=0}^{k} \delta \frac{\delta_0^{2^i}}{\alpha\delta^2\gamma} \le \frac{\delta_0}{\alpha\delta\gamma(1-\delta_0)} \le r,$$

and for $g(x_{k+1})$ we obtain

$$\|g(x_{k+1}) - x_0\| \le \|g(x_{k+1}) - g(x_0)\| + \|g(x_0) - x_0\| \le$$

$$\le \|[x_0, x_{k+1}; g]\| \cdot \|x_{k+1} - x_0\| + |\lambda| \cdot \|F(x_0)\| \le$$

$$\le \frac{\delta_0}{\alpha\delta(1-\delta_0)} + \frac{|\lambda|\,\delta_0}{\alpha\delta^2\gamma} = \frac{\delta_0}{\alpha\delta}\left(\frac{1}{1-\delta_0} + \frac{|\lambda|}{\delta\gamma}\right) \le r.$$

It can be easily seen that $(x_k)_{k \ge 0}$ is a Cauchy sequence, and hence it converges.

The stated evaluation of the error can be obtained from

$$\|x_{k+m} - x_k\| \le \frac{\delta_0^{2^k}}{\alpha\delta\gamma(1-\delta_0)}$$

for $m \to \infty$. The theorem is proved.

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Received August 15, 1996

"Tiberiu Popoviciu" Institute of Numerical Analysis
P.O. Box 68, 3400 Cluj-Napoca 1
Romania
e-mail: *ecatinas@ictp.math.ubbcluj.ro*
pavaloiu@ictp.math.ubbcluj.ro