REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome XXVII, N° 1, 1998, pp. 47–58

ANTIPROXIMINAL SETS IN THE BANACH SPACE $C(\omega^k; X)$

S. COBZAŞ

For a normed space X, a nonvoid subset M of X and an element $x \in X$ denote by $d(x, M) = \inf \{\|x - y\| : y \in M\}$ the *distance* from x to M and by $P_M(x) = \{y \in M : \|x - y\| = d(x, M)\}$ the set of all *nearest points* to x in M. The set M is called *proximinal* if $P_M(x) \neq \emptyset$ for all $x \in X$ and *antiproximinal* if $P_M(x) = \emptyset$, for all $x \in X \setminus M$. (Observe that $P_M(y) = \{y\}$, for all $y \in M$.)

Klee [19] denoted by N_1 the class of all normed spaces containing an antiproximinal closed convex set and by N_2 the class of all normed spaces containing an antiproximinal bounded closed convex set. Using James' characterization of reflexivity in terms of support functionals of the unit ball, V. Klee *loc. cit.* showed that a Banach space belongs to the class N_1 if and only if it is nonreflexive. The first example of a Banach space of class N_2 was found by M. Edelstein and A. C. Thompson [13] – the Banach space c_0 contains an antiproximinal bounded symmetric closed convex body. By a *convex body* we mean a convex set with nonvoid interior. In [8] it was shown that the Banach space c belongs to the class N_2 too, and this property is shared by any Banach space of continuous functions isomorphic to c_0 [9]. The existence of bounded closed antiproximinal convex sets in more general spaces of continuous functions was proved by V. P. Fonf [15]. Recently V. S. Balaganskii [3] has proved the existence of bounded antiproximinal convex bodies in any Banach space $C(T)$, for an arbitrary compact Hausdorff space T. By a result of D. Amir [1], a Banach space $C(T)$ of real-valued continuous functions on a compact Hausdorff space T is isomorphic to c_0 if and only if $C(T)$ is isometrically isomorphic to a space $C(\omega^k n)$ of continuous functions on the interval $[1, \omega^k n]$ of ordinal numbers, where ω denotes the first infinite ordinal.

The aim of the present paper is to extend the result from [9] to the vector-valued case. Similar results for the spaces $c_0(X)$ and $c(X)$ were obtained in [10] and [11].

THEOREM 1. *If X is a non-trivial Banach space, then the Banach space $C(\omega^k n; X)$ of all X-valued functions on the ordinal interval $[1, \omega^k n]$ contains an antiproximinal bounded symmetric closed convex body.*

Classification: 41A65, 46B20.

We shall consider only real Banach spaces and we agree to call a bounded symmetric closed convex body a *convex cell*. Concerning the properties of ordinal numbers, we shall follow the treatise [24], with the difference that in the normal Cantor expansion of a countable ordinal $\alpha, \alpha=\omega^{k_1}n_1+\ldots+\omega^{k_p}n_p, \omega>k_1>\ldots k_p$, we admit the possibility $n_i=0$, meaning that the corresponding term misses, e.g., $\omega^3 0+\omega^2 3+\omega 0+5=\omega^2 3+5$. We also adopt the convention $w^0=1$ and we shall denote by $\mathbb{N}=\{0,1,2,\ldots\}$ the set of natural numbers and by $\mathbb{N}^+=\{1,2,\ldots\}=[1,\omega)$ the set of positive natural numbers.

For the properties of topological spaces of ordinal numbers and of Banach spaces of continuous functions defined on intervals of ordinals we refer to [23]. For an ordinal number α, we denote by $C(\alpha;X)$ the Banach space (with the usual sup-norm) of all *X*-valued continuous functions defined on the interval $[1,\alpha]$ of ordinal numbers. It is well known that, equipped with the interval topology, $[1,\alpha]$ is a compact Hausdorff space (see [23, p. 151]). The Banach space $C(\alpha;\mathbb{R})$ will be denoted simply by $C(\alpha)$. The isomorphic classification of Banach spaces of type $C(\alpha)$ was given by C. Bessaga and A. Pelczynski [4] for countable ordinals and by S. P. Gul'ko and A. V. Os'kin [16], in general. The author is unaware whether some similar results are available in the vector-valued case.

The proof of Theorem 1 will proceed in several steps and it is different and, to some extent, simpler than that given in [9] for the scalar case. The main innovation consists in the use of an explicit form of an isomorphism between $c_0(X)$ and $C(\omega^k;X)$, inspired from the construction of the isomorphism between c_0 and c, given in [28, p. 55].

We show first that it is sufficient to prove Theorem 1 in the case $C(\omega^k;X)$.

LEMMA 1. *If the Banach space $C(\omega^k;X)$ contains an antiproximinal convex cell, then the Banach space $C(\omega^k n;X)$ also contains an antiproximinal convex cell, for every $n\in\mathbb{N}^+$.*

Proof. Let $\Delta=[1,\omega^k n]$ and $\Delta_i=[\omega^k(i-1)+1,\omega^k i]$, $i=1,\ldots,n$. Put $Y_i=\{x\chi_{\Delta_i}: x\in C(\Delta;X)\}$, where χ_{Δ_i} denotes the characteristic function of the set Δ_i. It is obvious that Y_i is linearly isometric to $C(\Delta_i;X)$ and, since Δ_i is homeomorphic to $[1,\omega^k]$, it follows that $C(\Delta_i;X)$ is linearly isometric to $C(\omega^k;X)$ for all $i=1,\ldots,n$. Since every $x\in C(\Delta;X)$ can be uniquely written in the form $x=x_1+\ldots+x_n$, with $x_i=x\chi_{\Delta_i}\in Y_i, i=1,\ldots,n$, it follows that $C(\Delta;X)$ is the direct algebraic sum of the subspaces Y_i. The equalities

(1) $$\|x\|=\max_{\alpha\in\Delta}\|x(\alpha)\|=\max_{1\le i\le n}\max_{\alpha\in\Delta_i}\|x_i(\alpha)\|$$

show that this sum is topological too, i.e.,

(2) $$C(\Delta;X)=Y_1\oplus\ldots\oplus Y_n.$$

Now, since the space Y_i is linearly isometric to $C(\omega^k;X)$, it contains an antiproximinal convex cell V_i. We shall show that the convex cell $V=V_1+\ldots+V_n$ is antiproximinal in $C(\Delta;X)$. Indeed, let $x=x_1=\ldots=x_n, x_i=x\chi_{\Delta_i}$, be an element in $C(\omega^k n;X)\setminus V$ and let $y=y_1+\ldots+y_n, y_i=y\chi_{\Delta_i}\in V_i, i=1,\ldots,n$. Put $N_1=\{i:1\le i\le n, \|x_i-y_i\|=\|x-y\|\}$ and $N_2=\{1,\ldots,n\}\setminus N_1$. It is obvious that $N_1\ne\emptyset$. If $i\in N_1$ is such that $x_i\notin V_i$, then, since V_i is antiproximinal in Y_i, there exists $y_i'\in V_i$ such that $\|x_i-y_i'\|<\|x_i-y_i\|$. If $i\in N_1$ and $x_i\in V_i$, then $y_i'=2^{-1}(x_i+y_i)\in V_i$ and $\|x_i-y_i'\|=2^{-1}\|x_i-y_i\|<\|x_i-y_i\|$. Letting $y_i'\in y_i$ for $i\in N_2$, it follows that the element $y'=y_1'+\ldots+y_n'\in V$ verifies

$$\|x-y'\|=\max\{\max_{i\in N_1}\|x_i-y_i'\|, \max_{i\in N_2}\|x_i-y_i'\|\}<\|x-y\|,$$

showing that x has no nearest points in V. □

Let $c_0(X)$ be the Banach space of all sequences $x:\mathbb{N}^+\to X$ such that $\lim_i x(i)=0$ normed by

(3) $$\|x\|=\sup_{i\in\mathbb{N}^+}\|x(i)\|.$$

In order to avoid a tedious notation (and taking into account Halmos' advice [17, p. 42]), in what follows we shall give some proofs only in the case $k=3$. The general case can be handled similarly.

LEMMA 2. *The Banach spaces $C(\omega^k;X)$ and $c_0(X)$ are linearly isomorphic.*

Proof. Let $k=3$. We shall identify the space $c_0(X)$ with the space $c_0(\mathbb{N}^3;X)$ of all functions $x:\mathbb{N}^3\to X$ such that the set $\{\lambda\in\mathbb{N}^3:\|x(\lambda)\|\ge\varepsilon\}$ is finite for every $\varepsilon>0$. For $x\in C(\omega^3;X)$ define $Hx:\mathbb{N}^3\to X$ by $Hx=y$, where

(4a) $$y(0,0,0)=x(\omega^3),$$

(4b) $$y(m,0,0)=x(\omega^2 m)-x(\omega^3),\quad m\in\mathbb{N}^+,$$

(4c) $$y(m,n,0)=x(\omega^2 m+\omega n)-x(\omega^2(m+1)),\quad m\in\mathbb{N},\ n\in N^+,$$

(4d) $$y(m,n,k)=x(\omega^2 m+\omega n+k)-x(\omega^2 m+\omega(n+1)),\quad m,n\in\mathbb{N},\ k\in\mathbb{N}^+.$$

First prove that $y \in c_0(\mathbb{N}^3; X)$. For $(m,n,k) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$ put $t_{m,n,k} = -2^{-m-1} - 2^{-m-n-2} - 2^{-m-n-k-2}$, $t_{0,0,0} = 0$ and let $T = \{t_{m,n,k} : (m,n,k) \in \mathbb{N}^3\}$. It follows that the application $h : [1, \omega^3] \to T$, defined by $h(\omega^3) = 0$ and $h(\omega^2 m + \omega n + k) = t_{m,n,k}$, for $(m,n,k) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$, is a (strictly increasing) homeomorphism between the compact spaces $[1, \omega^3]$ and T. Consequently, the topology of $[1, \omega^3]$ is generated by the metric

$$\rho(\alpha, \beta) = |h(\alpha) - h(\beta)| \tag{5}$$

for $\alpha, \beta \in [1, \omega^3]$.

Let $\varepsilon > 0$ be given. By the uniform continuity of the function x there exists a real number $\delta > 0$ such that

$$\|x(\alpha) - x(\beta)\| < \varepsilon \tag{6}$$

for all $\alpha, \beta \in [1, \omega^3]$ with $|h(\alpha) - h(\beta)| < \delta$. Choose $m_0, n_0, k_0 \in \mathbb{N}^+$ such that $2^{-m_0} < \delta, 2^{-n_0} < \delta, 2^{-k_0} < \delta$. Since $|h(\omega^2 m) - h(\omega^3)| = 2^{-m}$, $m \in \mathbb{N}^+$, $|h(\omega^2 m + \omega n) - h(\omega^2(m+1))| = 2^{-m-n-1}$, for $m \in \mathbb{N}$ and $n \in \mathbb{N}^+$, and $|h(\omega^2 m + \omega n + k) - h(\omega^2 m + \omega(n+1))| = 2^{-m-n-k-2}$ for $m, n \in \mathbb{N}$ and $k \in \mathbb{N}^+$ the relations (4) and (6) yield

$$\{(m,n,k) \in \mathbb{N}^3 : |y(m,n,k)| \geq \varepsilon\} \subseteq \{m,n,k) \in \mathbb{N}^3 : m \leq k_0, n \leq n_0, k \leq k_0\}.$$

Therefore $y \in c_0(\mathbb{N}^3; X)$.

It is obvious that the above defined operator $H : C(\omega^3; X) \to c_0(\mathbb{N}^3; X)$ is linear and, because $\|Hx\| \leq 2\|x\|$, it is also continuous. Since the equations (4) can be uniquely solved with respect to x, the operator H is a bijection and its inverse $G : c_0(\mathbb{N}^3; X) \to C(\omega^3; X)$ is given by $x = Gy$, where

$$x(\omega^3) = y(0,0,0), \tag{7}$$

$$x(\omega^2 m) = y(m,0,0) + y(0,0,0), \quad m \in \mathbb{N}^+,$$

$$x(\omega^2 m + \omega n) = y(m,n,0) + y(m+1,0,0) + y(0,0,0), \quad m \in \mathbb{N}, \quad n \in \mathbb{N}^+,$$

$$x(\omega^2 m + \omega n + k) = y(m,n,k) + y(m,n+1,0) + y(0,0,0), \quad m,n \in \mathbb{N}, \quad k \in \mathbb{N}^+.$$

It follows $\|Gy\| \leq 4\|y\|$, for all $y \in c_0(\mathbb{N}^3; X)$, implying the continuity of G. □

Now we shall construct a special isomorphism A of $C(\omega^3; X)$ onto itself in the following way: For an element $x \in C(\omega^3; X)$ define $Ax : [1, \omega^3] \to X$ by the formulae

$$Ax(\omega^3) = x(\omega^3) + 2^{-3} \sum_{1 \leq i < \omega} (-2)^{-i} x(\omega^2 i); \tag{8a}$$

$$Ax(\omega^2 m) = x(\omega^2 m) + 2^{-3} \sum_{1 \leq i \leq m-1} (-2)^{-i} x(\omega^2 i) + 2^{-m-2} \sum_{1 \leq i < \omega} (-2)^{-i} x(\omega^2(m-1) + \omega i), \quad m \in \mathbb{N}^+; \tag{8b}$$

$$Ax(\omega^2 m + \omega n) = x(\omega^2 m + \omega n) + 2^{-3} \sum_{1 \leq i \leq m} (-2)^{-i} x(\omega^2 i) + 2^{-m-3} \sum_{1 \leq i \leq n-1} (-2)^{-i} x(\omega^2 m + \omega i) + 2^{-m-n-2} \sum_{1 \leq i < \omega} (-2)^{-i} x(\omega^2 m + \omega(n-1) + 2i - 1), \quad m \in \mathbb{N}, \quad n \in \mathbb{N}^+; \tag{8c}$$

$$Ax(\omega^2 m + \omega n + k) = x(\omega^2 m + \omega n + k) + 2^{-3} \sum_{1 \leq i \leq m} (-2)^{-i} x(\omega^2 i) + 2^{-m-3} \sum_{1 \leq i \leq n} (-2)^{-i} x(\omega^2 m + \omega i) + 2^{-m-n-3} \sum_{1 \leq i \leq k} (-2)^{-i} x(\omega^2 m + \omega n + 2i - 1) + 2^{-m-n-k-2} \sum_{1 \leq i < \omega} (-2)^{-i} x(\omega^2 m + \omega n + 2^k(2i-1)), \quad m,n \in \mathbb{N}, \quad k \in \mathbb{N}^+. \tag{8d}$$

(We adopt the convention $\sum_{i \in \emptyset} a_i = 0$.)

LEMMA 3. *The application A defined by the formulae* (8) *is an isomorphism of $C(\omega^3; X)$ onto itself.*

Proof. A careful examination of the formulae (8) shows that $Ax \in C(\omega^3; X)$ for $x \in C(\omega^3; X)$. This follows from the relations $\lim_{m \to \omega} (\omega^2 m + \omega n + k) = \omega^3$, $\lim_{n \to \omega} (\omega^2 m + \omega n + k) = \omega^2(m+1)$, $\lim_{k \to \omega} (\omega^2 m + \omega n + k) = \omega^2 m + \omega(n+1)$, the continuity of the function x and the definition of Ax given by (8). The linearity of A is obvious and, by (8)

$$\|Ax(\alpha\| \leq \|x\| + 4 \cdot 2^{-3}\|x\| = (3/2)\|x\|,$$

for all $\alpha \in [1, \omega^3]$, implying

(9) $$\|Ax\| \le (3/2)\|x\|,$$

for all $x \in C(\omega^3; X)$, which is equivalent to the continuity of A.

Now, for $x \in C(\omega^3; X)$ choose $\alpha \in [1, \omega^3]$ such that

$$\|x(\alpha)\| = \max\{\|x(\beta)\| : \beta \in [1, \omega^3]\}.$$

Taking into account all the possibilities appearing in formulae (8), we conclude that

$$\|Ax\| \ge \|Ax(\alpha\| \ge \|x(\alpha)\| - 2^{-1}\|x\| = (1/2)\|x\|.$$

Therefore

(10) $$\|Ax\| \ge (1/2)\|x\|.$$

The inequalities (9) and (10) show that A is an isomorphism of $C(\omega^3; X)$ onto $C(\omega^3; X)$, which ends the proof of Lemma 3. □

The key tools used in the proof of Theorem 1 will be the following two results concerning the support functionals of convex sets in Banach spaces (Lemmas 5 and 6 below).

Let X be a Banach space, X^* its conjugate and M a nonvoid closed convex subset of X. A functional $f \in X^*$ is said to *support* M (at x) if there exists $x \in M$ such that $f(x) = \inf f(M)$ or $f(x) = \sup f(M)$. A functional $f \in X^*$ supports the closed unit ball B_X of X if and only if there exists $x \in B_X$ such that $f(x) = \|f\|$. If $f \ne 0$, then every $x \in B_X$ satisfying this equality must be of norm one, i.e., $\|x\| = 1$. We shall denote by $\mathscr{S}(M)$ the set of all support functionals of the set M.

The following characterization of antiproximinal sets appears in [11]. Other characterizations were given by A.-M. Precupanu and T. Precupanu [22].

LEMMA 4. *A nonvoid closed convex subset M of a Banach space X is antiproximinal if and only if*

(11) $$\mathscr{S}(M) \cap \mathscr{S}(B_X) = \{0\},$$

where B_X denotes the closed unit ball of X.

If X,Y are Banach spaces and $A : X \to Y$ is an isomorphism then its conjugate $A^* : Y^* \to X^*$ is an isomorphism, too, and $(A^*)^{-1} = (A^{-1})^*$ (see [12, Lemma VI.3.7]). The support functionals of a set $M \subseteq X$ and of its image $A(M)$ are related as follows:

LEMMA 5. [13, Lemma 1]. *Let X, Y be two Banach spaces and $A : X \to Y$ an isomorphism. If M is a nonvoid closed convex subset of X then*

(12) $$\mathscr{S}(M) = A^*(\mathscr{S}(A(M))).$$

More exactly,

(13) $$g \in \mathscr{S}(A(M)) \Leftrightarrow A^* g \in \mathscr{S}(M).$$

The proof of the existence of an antiproximinal convex cell in $C(\omega^3; X)$ will be based on Lemma 4, so that we need some information about the behaviour of the support functionals of the unit ball of $C(\omega^3; X)$.

The characterization of the support functionals of the unit ball of $C(T) = C(T; \mathbb{R})$, T – a compact Hausdorff space, was given by S. I. Zukhovickij [27] in the metric case and by R. R. Phelps [21], in general. The vector-valued case was considered by V. L. Chakalov [5] and L. P. Vlasov [26].

Let ν be a countable ordinal and let $\Delta = [1, \nu]$. The dual space of $C(\Delta; X)$ can be identified with the Banach space $l^1(\Delta; X^*)$ of functions $f : \Delta \to X^*$, $f = (f_\alpha : \alpha \in \Delta)$, such that

(14) $$\|f\| := \sum_{\alpha \in \Delta} \|f_\alpha\| < \infty.$$

The duality between $C(\Delta; X)$ and $l^1(\Delta; X^*)$ is given by the formula

(15) $$f(x) = \sum_{\alpha \in \Delta} f_\alpha(x(\alpha)),$$

for $f = (f_\alpha : \alpha \in \Delta)$ in $l^1(\Delta; X^*)$ and $x = (x(\alpha) : \alpha \in \Delta)$ in $C(\Delta; X)$.

Denoting by B_C the closed unit ball of $C(\Delta; X)$, we have

LEMMA 6. a) *If the functional $f = (f_\alpha : \alpha \in \Delta) \in l^1(\Delta; X^*)$, $f \ne 0$, supports the unit ball B_C of $C(\Delta; X)$ at $x \in B_C$, then $f_\alpha(x(\alpha)) = \|f_\alpha\|$ for all $\alpha \in \Delta$ and $\|x(\alpha)\| = 1$, for all $\alpha \in \Delta$ such that $f_\alpha \ne 0$.*

b) *Let $\gamma \in \Delta$ be a limit ordinal and suppose that $(\alpha_k : k \in \mathbb{N}^+)$ and $(\beta_k : k \in \mathbb{N}^+)$ are two strictly increasing sequences in Δ such that $\lim_k \alpha_k = \lim_k \beta_k = \gamma$ and $\alpha_k \ne \beta_l$, for all $k, l \in \mathbb{N}^+$. Suppose further that two sequences (a_k) and (b_k) of strictly positive real numbers and a functional $h \in X^*$, $h \ne 0$, are given. If $f \in l^1(\Delta; X^*)$ is such that $f_{\alpha_k} = a_k h$ and $f_{\beta_k} = -b_k h$, for all $k \in \mathbb{N}^+$, then $f \notin \mathscr{S}(B_C)$.*

Proof. a) Since $f_\alpha(x(\alpha)) \le \|f_\alpha\| \cdot \|x(\alpha)\|$, for all $\alpha \in \Delta$, it follows

$$\sum_{\alpha\in\Delta} \|f_\alpha\| = \|f\| = f(x) = \sum_{\alpha\in\Delta} f_\alpha(x(\alpha)) \le$$

$$\le \sum_{\alpha\in\Delta} \|f_\alpha\| \cdot \|x(\alpha)\| \le \sum_{\alpha\in\Delta} \|f_\alpha\|,$$

implying $f_\alpha(x(\alpha)) = \|f_\alpha\|$, for all $\alpha \in \Delta$, and $\|x(\alpha)\| = 1$ whenever $f_\alpha \ne 0$.

b) Suppose that $h \in X^*$, $\alpha_k, \beta_k, \gamma \in \Delta, a_k, b_k > 0$ and $f \in l^1(\Delta; X^*)$ fulfill the hypotheses of the lemma and suppose, on the contrary, that there exists $x = (x(\alpha) : \alpha \in \Delta)$ in B_C such that $f(x) = \|f\|$. Taking into account the first assertion of the lemma, we obtain

$$a_k \|h\| = \|f_{\alpha_k}\| = a_k h(x(\alpha_k))$$

and

$$b_k \|h\| = \|f_{\beta_k}\| = -b_k h(x(\beta_k)),$$

implying $h(x(\alpha_k)) = \|h\|$ and $h(x(\beta_k)) = -\|h\|$, for all $k \in \mathbb{N}^+$. Since both (α_k) and (β_k) tend to γ for $k \to \infty$, these two equalities yield, for $k \to \infty$, the contradiction $h(x(\gamma)) = \|h\| > 0$ and $h(x(\gamma)) = -\|h\| < 0$. □

Now let Λ be an infinite countable set and let $c_0(\Lambda; X)$ denote the Banach space of all functions $x : \Lambda \to X$ such that the set $\{\lambda \in \Lambda : \|x(\lambda)\| \ge \varepsilon\}$ is finite, for every $\varepsilon > 0$. The norm on $c_0(\Lambda; X)$ is given by

(16) $$\|x\| = \max\{\|x(\lambda)\| : \lambda \in \Lambda\}.$$

The conjugate space of $c_0(\Lambda; X)$ is the Banach space $l^1(\Lambda; X^*)$ of all functions $f : \Lambda \to X^*, f = (f_\lambda : \lambda \in \Lambda)$, such that

(17) $$\|f\| := \sum_{\lambda\in\Lambda} \|f_\lambda\| < \infty.$$

The duality between $c_0(\Lambda; X)$ and $l^1(\Lambda; X^*)$ is given by

(18) $$f(x) = \sum_{\lambda\in\Lambda} f_\lambda(x_\lambda)),$$

for $f = (f_\lambda : \lambda \in \Lambda)$, in $l^1(\Lambda; X^*)$ and $x = (x(\lambda) : \lambda \in \Lambda)$ in $c_0(\Lambda; X)$.

A characterization of support functionals of the unit ball of $c_0(\Lambda; X)$ is given in the following.

LEMMA 7. *A functional $f = (f_\lambda : \lambda \in \Lambda) \in l^1(\Lambda; X^*), f \ne 0,$ supports the closed unit ball B_{c_0} of $c_0(\Lambda; X)$ if and only if there exists a nonvoid finite subset Γ of Λ such that $f_\lambda = 0$ for $\lambda \in \Lambda \setminus \Gamma$ and $f_\lambda \in \mathscr{S}(B_X) \setminus \{0\}$ for $\lambda \in \Gamma$, where B_X denotes the closed unit ball of X.*

Proof. Let $f \in \mathscr{S}(B_{c_0})$, $f \ne 0$, and let $x \in B_{c_0}$ be such that $f(x) = \|f\|$. Reasoning as in the proof of Lemma 6a), we obtain $f_\lambda(x(\lambda)) = \|f_\lambda\| \cdot \|x(\lambda)\| = \|f_\lambda\|$, for all $\lambda \in \Lambda$, implying $f_\lambda = 0$ for all $\lambda \in \Lambda$ such that $\|x(\lambda)\| < 1$. Since, by the definition of the space $c_0(\Lambda; X)$ the set $\Gamma = \{\lambda \in \Lambda : \|x(\lambda)\| = 1\}$ is finite, the necessity part of the lemma is proved.

Conversely, let Γ be a nonvoid finite subset of Λ and let $f = (f_\lambda : \lambda \in \Lambda)$ be an element in $l^1(\Lambda; X^*)$ such that $f_\lambda = 0$ for $\lambda \in \Lambda \setminus \Gamma$ and $f_\lambda \in \mathscr{S}(B_X) \setminus \{0\}$, if $\lambda \in \Gamma$. If $x_\lambda \in X, \|x_\lambda\| = 1$, is such that $f_\lambda(x_\lambda) = \|f_\lambda\|$, for $\lambda \in \Gamma$, and $x : \Lambda \to X$ is given by $x(\lambda) = x_\lambda$, for $\lambda \in \Gamma$, and $x(\lambda) = 0$, for $\lambda \in \Lambda \setminus \Gamma$ then $f(x) = \|f\|$, showing that $f \in \mathscr{S}(B_{c_0})$. □

Now we are ready to proceed to:

Proof of Theorem 1. Take again $k = 3$ and denote by B_C and B_{c_0} the closed unit balls of $C(\omega^3; X)$ and $c_0(\mathbb{N}^3; X)$ respectively. Let $H : C(\omega^3; X) \to c_0(\mathbb{N}^3; X)$ be the isomorphism from Lemma 2 (defined by the formulae (4)) and let A be the isomorphism of $C(\omega^3; X)$ onto itself given by the formulae (8) (see Lemma 4). It follows that the set

(19) $$V = (HA)^{-1}(B_{c_0})$$

is a convex cell (i.e., a bounded symmetric closed convex body) in $C(\omega^3; X)$ and let us show that V is an antiproximinal subset of $C(\omega^3; X)$. By Lemma 4, this is equivalent to

(20) $$\mathscr{S}(V) \cap \mathscr{S}(B_C) = \{0\}.$$

But, by (19), $B_{c_0} = HA(V)$, implying

(21) $$\mathscr{S}(B_{c_0}) = \mathscr{S}(HA(V)),$$

which, by Lemma 5, gets

(22) $$\mathscr{S}(V) = \{(HA)^* f : f \in \mathscr{S}(B_{c_0})\}.$$

It follows that relation (20) will be a consequence of the following implication

(23) $$f\in\mathcal{S}(B_{c_0})\setminus\{0\}\Rightarrow (HA)^* f\notin\mathcal{S}(B_C).$$

In order to prove (23), suppose that $f=(f_\lambda:\lambda\in\mathbb{N}^3)\in l^1(\mathbb{N}^3;X^*)$, $f\neq 0$, is a support functional of the unit ball B_{c_0} of $c_0(\mathbb{N}^3;X)$. By Lemma 7, there exists a finite subset, say $\Gamma=\{\lambda_1,\dots,\lambda_p\}$ of $\mathbb{N}^3$, such that $f_\lambda=0$, for $\lambda\in\mathbb{N}^3\setminus\Gamma$ and $f_\lambda\in\mathcal{S}(B_X)\setminus\{0\}$ for $\lambda\in\Gamma$. It follows

(24) $$(HA)^*f(x)=f(HAx)=\sum_{j=1}^{p} f_{\lambda_j}((HAx)(\lambda_j)),$$

for all $x\in C(\omega^3;X)$. Now, taking into account the formulae (4) defining the isomorphism H, we obtain:

(25a) $$(HAx)(0,0,0)=Ax(\omega^3);$$

(25b) $$(HAx)(m,0,0)=Ax(\omega^3 m)-Ax(\omega^3),\quad m\in\mathbb{N}^+;$$

(25c) $$(HAx)(m,n,0)=Ax(\omega^3 m+\omega n)-Ax(\omega^2(m+1)),\quad m\in\mathbb{N},\quad n\in\mathbb{N}^+,$$

(25d) $$(HAx)(m,n,k)=Ax(\omega^3 m+\omega n+k)- - Ax(\omega^2 m+\omega(n+1)),\quad m,n\in\mathbb{N},\quad k\in\mathbb{N}^+.$$

In order to show that $(HA)^*f\notin\mathcal{S}(B_C)$, we shall resort to Lemma 6b). Let $\lambda_j=(m_j,n_j,k_j)\in\mathbb{N}^3$ and $\phi(\lambda_j)=\omega^3 m_j+\omega n_j+k_j$ if $\lambda_j\neq(0,0,0)$ and $\phi(0,0,0)=\omega^3$, for $j=1,\dots,p$. Let also $\Lambda_1=\{\phi(\lambda_j):k_j\geq 1\}$, $\Lambda_2=\{\phi(\lambda_j):k_j=0,n_j\geq 1\}$, and $\Lambda_3=\{\phi(\lambda_j):n_j=k_j=0,m_j\geq 1\}$. If $\Lambda_1\neq\emptyset$ pick $j\in\{1,\dots,p\}$ such that $\phi(\lambda_j)=\max\Lambda_1$. Taking into account the formulae (25d), (24) and (8d), we get

$$((HA)^*f)_{\alpha_i}=(-1)^i 2^{-m_j-n_j-k_j-i}\cdot f_{\lambda_j},$$

for sufficiently large $i\in[1,\omega]$, where $\alpha_i=\omega^2 m_j+\omega n_j+2^{k_j}(2i-1)\to\omega^2 m_j+ +\omega(n_j+1)$, for $i\to\omega$. If $\Lambda_1=\emptyset$, $\Lambda_2=\emptyset$ and $j\in\{1,\dots,p\}$ is such that $\phi(\lambda_j)=\max\Lambda_2$, then, by (25c), (24) and (8c), one obtains

$$((HA)^*f)_{\alpha_i}=(-1)^i 2^{-m_j-n_j-i-2}\cdot f_{\lambda_j},$$

for all $i\in[1,\omega]$ sufficiently large, where $\alpha_i=\omega^2 m_j+\omega(n_j-1)+2i-1\to\omega^2 m_j+\omega n_j$, for $i\to\omega$. If $\Lambda=\Lambda_2=\emptyset$, $\Lambda_3\neq\emptyset$ and $\phi(\lambda_j)=\max\Lambda_3$, then, by (25b), (24) and (8b),

$$((HA)^*f)_{\alpha_i}=(-1)^i 2^{-m_j-i-2}\cdot f_{\lambda_j},$$

for all $i\in[1,\omega]$ sufficiently large, where $\alpha_i=\omega^3(m_j-1)+\omega i\to\omega^2 m_j$, for $i\to\omega$.

Finally, if $\Lambda_1=\Lambda_2=\Lambda_3=\emptyset$, then $\Gamma=\{(0,0,0)\}$ and, by (25a), (24) and (8a), we obtain

$$((HA)^*f)_{\alpha_i}=(-1)^i 2^{-i-3}\cdot f_{(0,0,0)},$$

for all $i\in[1,\omega[$, where $\alpha_i=\omega^2 i\to\omega^3$, for $i\to\omega$.

It follows that in all these cases we can apply Lemma 6b) to conclude that $(HA)^*f$ is not in $\mathcal{S}(B_C)$.

Theorem 1 is completely proved.

BIBLIOGRAPHY

1. D. Amir, *Continuous function spaces with the separable extension property*, Bull. Res. Council Israel **10F** (1962), 163–164.
2. V. S. Balaganskii, *An antiproximinal set in a strictly convex space with Fréchet differentiable norm*, East J. Approx., **2** (1996), 169–176.
3. V. S. Balaganskii, *Antiproximinal sets in spaces of continuous functions*, Matem. Zametki **60** (1996), 643–657 (in Russian).
4. C. Bessaga and A. Pelczynski, *Spaces of continuous functions (IV). (On isomorphical classification of spaces C(S))*, Studia Math. **19** (1960), 53–62.
5. V. L. Chakalov, *Extremal elements in some normed spaces*, Comptes Rendus Acad. Bulgare des Sciences **36** (1983), 173–176.
6. S. Cobzaş *Very non-proximinal sets in c_0*, Rev. Anal. Numér. Théorie Approximation **2** (1973), 137–141. (in Romanian).
7. S. Cobzaş, *Antiproximinal sets in some Banach spaces*, Math. Balkanica **4** (1974), 79–82.
8. S. Cobzaş, *Convex antiproximinal sets in the spaces c_0 and c*, Matem. Zametki **17** (1975), 449–457 (in Russian).
9. S. Cobzaş, *Antiproximinal sets in Banach spaces of continuous functions*, Rev. Anal. Numér. Théorie Approximation **5** (1976), 127–143.
10. S. Cobzaş, *Antiproximinal sets in Banach spaces of c_0-type*, Rev. Anal. Numér. Théorie Approximation **7** (1978), 141–145.
11. S. Cobzaş, *Antiproximinal sets in the Banach space c(X)*, Commentationes Math. Univ. Carolinae **38** (1997), 247–253.
12. N. Dunford and J. T. Schwartz, *Linear Operators I. General Theory*, Interscience, New York 1958.

13. M. Edelstein and A. C. Thompson, *Some results on nearest points and support properties of convex sets in* c_0, Pacific J. Math. **40** (1972), 553–560.
14. V. P. Fonf, *On antiproximinal sets in spaces of continuous functions on compacta*, Matem. Zametki **33** (1983), 549–558 (in Russian).
15. V. P. Fonf, *On strongly antiproximinal sets in Banach spaces*, Matem. Zametki **47** (1990), 130–136. (in Russian).
16. S. P. Gul'ko and A. V. Os'kin, *Isomorphic classification of spaces of continuous functions on totally ordered compacta*, Funkc. Analiz i ego Priloženija **9**, *1* (1975), 61–62.
17. P. R. Halmos, *How to Write Mathematics*, American Mathematical Society, 1973.
18. R. B. Holmes, *Geometric Functional Analysis and Its Applications*, Springer Verlag, Berlin–Heidelberg–New York, 1975.
19. V. Klee, *Remarks on Nearest Points in Normed Linear Spaces*, Proc. Colloq. Convexity, Copenhagen, 1965, p. 161–176, Copenhagen, 1967.
20. R. R. Phelps, *Subreflexive normed linear spaces*, Archiv der Math. **8** (1957), 444–450.
21. R. R. Phelps, *Some subreflexive Banach spaces*, Archiv. der Math. **10** (1959), 162–169.
22. A.-M. Precupanu and T. Precupanu, *A characterization of antiproximinal sets*, Analele St. Univ. Iaşi **33**, seria I, Matematica 1987, f. 2, 99–106.
23. Z. Semadeni, *Banach Spaces of Continuous Functions*, PWN Warszawa, 1965.
24. W. Sierpinski, *Cardinal and Ordinal Numbers*, PWN Warszawa, 1965.
25. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Ed. Academiei and Springer Verlag, Bucharest – Berlin 1970.
26. L. P. Vlasov, *Norm attaining functionals on the space* $C(Q, X)$, Matem. Zametki **61** (1997), 45–56.
27. S. I. Zukhovickij, *On minimal extensions of linear functionals in spaces of continuous functions*, Izvestija Akad. Nauk SSSR, Ser. Matem. **21** (1957), 409–422 (in Russian).
28. D. Werner, *Funktionalanalysis*, Springer Verlag, Berlin – Heidelberg – New York, 1995.

Received December 15, 1997

Faculty of Mathematics
"Babeş-Bolyai" University
Ro – 3400 Cluj-Napoca
Romania
E-mail: scobzas@math.ubbcluj.ro