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ON APPROXIMATION PROPERTIES OF STANCU-KANTOROVICH OPERATORS*

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1. INTRODUCTION

The Stancu polynomials [14], defined by

$$S_n^\alpha(f;x)=\sum_{k=0}^{n}\omega_{n,k}^\alpha(x)\,f\left(\frac{k}{n}\right),\quad x\in I:=[0,1], \tag{1.1}$$

where

$$\omega_{n,k}^\alpha(x)=\binom{n}{k}\frac{x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}},$$

$$x^{(k,-\alpha)}=x(x+\alpha)\dots(x+(k-1)\alpha),\quad \alpha\ge 0,$$

can be used for constructing a class of Stancu-Kantorovich polynomials [13]:

$$K_n^\alpha(f;x)=(n+1)\sum_{k=0}^{n}\omega_{n,k}^\alpha(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)\,\mathrm{d}t. \tag{1.2}$$

These types of parameter-dependent approximation methods represent interesting natural generalizations of the classical Bernstein-Kantorovich polynomials.

Further, the Stancu polynomials are related to Pólya polynomials often used in Computer Aided Geometric Design (CAGD). (See, e.g., [2], [4] and [5]). Indeed, they have many remarkable properties desirable in graphics, such as affine invariance, the convex hull property, nondegeneracy, interpolation of first and last control points, a recursive evaluation algorithm, a simple subdivision technique, an

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elementary symmetry property, a compact explicit formula, a two-term degree elevation formula and the variation diminishing property. For other properties interesting in approximation theory see, e.g., [3].

In the following, C, C_j $(j\in\mathbb{N})$ denote positive constants which can assume different values in different formulas.

By simple computations we can get explicit expressions for the first and second moments of K_n^α. Indeed, putting $\Omega_{i,x}(t)=(t-x)^i$, with $i\in N_0, n\ge 2$ and $t,x\in I$, we have

LEMMA 1.1. *Let K_n^α be given by* (1.2). *Then*

$$K_n^\alpha(\Omega_{0,x};x)=1,$$

$$K_n^\alpha(\Omega_{1,x};x)=\frac{1-2x}{2(n+1)},\tag{1.3}$$

$$K_n^\alpha(\Omega_{2,x};x)=x(1-x)\frac{n\frac{n\alpha+1}{\alpha+1}-1}{(n+1)^2}+\frac{1}{3(n+1)^2}.\tag{1.4}$$

Moreover, if

$$0\le\alpha\le\frac{C}{n},$$

with C a positive constant, it follows

$$K_n^\alpha(\Omega_{2,x};x)\le C_1\left\{\frac{1}{(n+1)^2}+\frac{\phi^2(x)}{n+1}\psi_n(x)\right\},\tag{1.6}$$

where

$$\psi_n(x)=\begin{cases}1, & x\in E_n\\ 0, & x\in I/E_n,\end{cases}$$

with $E_n:=\left[\frac{A}{n},1-\frac{A}{n}\right]$ and A an arbitrary but fixed positive number.

LEMMA 1.2. *For a fixed point $x_0\in I$ and $0\le\alpha\le\frac{C}{n}$, we have*

$$\lim_{n\to\infty}\frac{(n+1)^2(1+\alpha)}{n(n\alpha+1)-(\alpha+1)}K_n^\alpha(R;x_0)=0,\tag{1.7}$$

where

$$R(t):=\Omega_{2,x_0}(t)\,\rho(t-x_0),$$

and ρ is a bounded function with $\lim_{u\to 0}\rho(u)=0$, i.e.,

$$\forall\ \varepsilon>0\ \text{ there exists }\ \delta=\delta(\varepsilon):|\rho(t-x_0)|<\varepsilon,\quad\forall\ |t-x_0|<\delta,\tag{1.8}$$

and

$$\forall\ |t-x_0|\ge\delta,\quad |\rho(t-x_0)|\le B,\quad B:=\sup\{|\rho(t-x_0)|\}.\tag{1.9}$$

Proof. Let

$$K_n^\alpha(R(t);x_0)=K_n^\alpha(\Omega_{2,x_0}(t)\,\rho(t-x_0);x_0)=$$

$$=(n+1)\sum_{k=0}^{n}\omega_{n,k}^\alpha(x_0)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\Omega_{2,x_0}(t)\,\rho(t-x_0)\,dt=$$

$$=(n+1)\left\{\sum_{\left|\frac{k}{n}-x_0\right|<\delta}+\sum_{\left|\frac{k}{n}-x_0\right|\ge\delta}\right\}\omega_{n,k}^\alpha(x_0)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\Omega_{2,x_0}(t)\,\rho(t-x_0)\,dt=$$

$$=:S_1(x_0)+S_2(x_0).$$

Now, if $0\le\alpha\le\frac{C}{n}$, from (1.6) and (1.8) we get

$$|S_1(x_0)|\le\varepsilon|K_n^\alpha(\Omega_{2,x_0};x_0)|\le\varepsilon C_1[n^{-1}(\phi^2(x_0)\,\psi_n(x_0)+n^{-1})]\le\varepsilon Cn^{-1}.$$

On the other hand, since

$$\left(\frac{k+1}{n+1}-x_0\right)^2\le C\left[\left(\frac{k}{n}-x_0\right)^2+n^{-2}\right],$$

by (1.9) it follows

$$|S_2(x_0)|\le B\left|(n+1)\sum_{\left|\frac{k}{n}-x_0\right|\ge\delta}\omega_{n,k}^\alpha(x_0)\int_0^{\frac{1}{n+1}}\left(t+\frac{k}{n+1}-x_0\right)^2dt\right|\le$$

$$\le BC\left[\sum_{\left|\frac{k}{n}-x_0\right|\ge\delta}\omega_{n,k}^\alpha(x_0)\left[\left(\frac{k}{n}-x_0\right)^2+n^{-2}\right]\right]\le$$

$$\le BC\delta^{-2}\sum_{\left|\frac{k}{n}-x_0\right|\ge\delta}\omega_{n,k}^\alpha(x_0)\left(\frac{k}{n}-x_0\right)^4+BCn^{-2}.$$

Now, since [15, p. 56] for $x \in I$,

$$S_n^\alpha(\Omega_{4,x};x) \le C_2 \frac{\frac{1}{n}+\alpha}{n(1+\alpha)}(1+n\alpha),$$

with C_2 a positive constant independent of n and x_0, we get

$$|S_2(x_0)| \le BC\left[\delta^{-2} C_2 \frac{\frac{1}{n}+\alpha}{n(1+\alpha)}(1+n\alpha)+n^{-2}\right];$$

therefore, for $\varepsilon > 0$,

$$S_1(x_0)+S_2(x_0) \le C_3[\varepsilon n^{-1}+n^{-2}\delta^{-2}],$$

with C_3 a positive constant independent of n and x_0, from which the assertion follows. □

Now we can prove the following asymptotic relation of Voronovskaja type for K_n^α.

THEOREM 1.3. *Let $f \in C(I)$ be a bounded and twice differentiable function at a fixed point $x_0 \in I$. Then for $\alpha \le \frac{C}{n}$,*

(1.10) $$\lim_{n\to\infty} c_{n,\alpha}\left\{K_n^\alpha(f;x_0)-f(x_0)-f'(x_0)\frac{1-2x_0}{2(n+1)}\right\} = \frac{1}{2}\phi^2(x_0)f''(x_0),$$

with $c_{n,\alpha} = \dfrac{(n+1)^2\,(1+\alpha)}{n(n\alpha+1)-(1+\alpha)}$.

Proof. By expanding the function f by Taylor formula at a fixed point x_0, we get

$$f(t) = P(t)+R(t), \quad t \in I,$$

with

$$P(t) = \sum_{i=0}^{2} \frac{f^{(i)}(x_0)}{i!}\Omega_{i,x_0}(t), \quad \Omega_{i,x_0}(t) = (t-x_0)^i,$$

and

$$R(t) = \Omega_{2,x_0}(t)\,\rho(t-x_0),$$

where ρ is a function defined in (1.8) – (1.9). Then, from Lemma 1.1,

$$K_n^\alpha(f;x_0)\sum_{i=0}^{2}\frac{f^{(i)}(x_0)}{i!}K_n^\alpha(\Omega_{i,x_0};x_0) = f(x_0)+\frac{1}{2(n+1)}(1-2x_0)f'(x_0)+$$

$$+\frac{1}{2}f''(x_0)\left[\left(\frac{n(n\alpha+1)}{1+\alpha}-1\right)\frac{\phi^2(x_0)}{(n+1)^2}+\frac{1}{3(n+1)^2}\right]+K_n^\alpha(R;x_0).$$

Hence, if $0 \le \alpha \le \frac{C}{n}$, we have

$$\lim_{n\to\infty} c_{n,\alpha}\left\{K_n^\alpha(f;x_0)-f(x_0)-\frac{1}{2(n+1)}(1-2x_0)f'(x_0)\right\} =$$

$$=\frac{1}{2}\phi^2(x_0)f''(x_0)+\frac{1}{3}\lim_{n\to\infty}\frac{1+\alpha}{n(n\alpha+1)-(1+\alpha)}+\lim_{n\to\infty} c_{n,\alpha}K_n^\alpha(R;x_0).$$

Then, by (1.7) in Lemma 1.2 and since $\dfrac{1+\alpha}{n(n\alpha+1)-(1+\alpha)} \sim (n-1)^{-1}$, the assertion follows. □

Remark. In the case $\alpha = 0$ from (1.10) we find the classical result for Kantorovich operator

(1.11) $$\lim_{n\to\infty} n\{K_n^0(f;x_0)-f(x_0)\} = \frac{1}{2}(\phi^2(x_0)f'(x_0))'.$$

The following theorem shows that the order of approximation by K_n^α increases near the endpoints ± 1 of the interval I. To this aim, we recall the Lipschitz type maximal function $\tilde{f}_\beta$ of order β introduced in [6] and defined as

$$\tilde{f}_\beta(x) = \sup_{t\ne x,\, t\in I}\frac{|f(x)-f(t)|}{|x-t|^\beta}, \quad x \in I,\ \beta \in (0,1].$$

For further applications see also [7–9] and [11].

Denoting by $\omega(f;\delta)$ the usual modulus of continuity of f, we have

THEOREM 1.4. *Let $K_n^\alpha(f)$ be defined by (1.2). Then for $f \in C(I)$ and $0 \le \alpha \frac{1}{n}$,*

(1.12) $$|f(x)-K_n^\alpha(f;x)| \le \begin{cases} 2\omega\left(f;\sqrt{\frac{1}{n}\left(\frac{n\alpha+1}{\alpha+1}x(1-x)+\frac{1}{n}\right)}\right), \\ \tilde{f}_{\beta(x)}\left(\frac{n\alpha+1}{n(\alpha+1)}x(1-x)+\frac{1}{n^2}\right)^{\frac{\beta}{2}}. \end{cases}$$

Proof. From the estimates

$$|f(x)-f(t)| \le \left(1+\frac{1}{\delta}|x-t|\right)\omega(f;\delta)$$

and

$$|f(x)-f(t)|\le \tilde{f}_\beta(x)\,(K_n^\alpha(\Omega_{2,x};x))^{\frac{\beta}{2}},$$

working similarly as in [7–9] and [11], we get the assertion. □

Now we want to give direct approximation results for K_n^α operator. To this aim, putting $\|\cdot\|=\|\cdot\|_\infty$ the usual supremum norm on I, we need the following

LEMMA 1.5. *Let $K_n^\alpha(f)$ be defined by* (1.2) *and $\phi(x)=\sqrt{x(1-x)}$. Then for $f\in C^2(I)$ we have*

$$|f(x)-K_n^\alpha(f;x)|\le \frac{C}{n}\left(\|f'\|_\infty+\frac{(n\alpha+1)}{\alpha+1}\|\phi^2 f''\|_\infty\right),\quad x\in I, \tag{1.13}$$

with C a positive constant independent of f, x and n.

Proof. From the second moment of K_n^α (formula (1.6), for $\phi^2(x)<\left(\frac{n(n\alpha+1)}{\alpha+1}\right)^{-1}$, we get

$$|f(x)-K_n^\alpha(f;x)|=\left|K_n^\alpha\left(\int_x^t f'(v)\,dv;x\right)\right|\le$$

$$\le\|f'\|_\infty\sqrt{K_n^\alpha(\Omega_{2,x};x)}\le C_1\frac{\|f'\|_\infty}{n}.$$

On the other hand, since

$$f(t)=f(x)+f'(x)\,\Omega_{1,x}(t)+\int_x^t \Omega_{1,v}(t)\,f''(v)\,dv,$$

and for $v\in[x,t]$ or $v\in[t,x]$ [1, p. 141]

$$\frac{|t-v|}{\phi^2(v)}\le\frac{|t-x|}{\phi^2(x)},$$

from (1.3) and (1.4) we have for $\phi^2(x)>\left(\frac{n(n\alpha+1)}{\alpha+1}\right)^{-1}$

$$|f(x)-K_n^\alpha(f;x)|\le\frac{1-2x}{2(n+1)}\|f'\|_\infty+\frac{\|\phi^2 f''\|_\infty}{\phi^2(x)}K_n^\alpha(\Omega_{2,x};x)\le$$

$$\le C\left\{\frac{1}{n}\|f'\|_\infty+\frac{n\alpha+1}{n(\alpha+1)}\|\phi^2 f''\|_\infty\right\},$$

that is (1.13). □

Now we can prove direct results for K_n^α operator. Indeed, letting

$$\omega_\phi^2(f,t)_\infty=\sup_{h\le t}\|\Delta_{h\phi}^2 f\|_\infty,\quad \phi(x)=\sqrt{x(1-x)},$$

the second modulus of smoothness of Ditzian-Totik [1], we have

THEOREM 1.6. *Let $K_n^\alpha(f)$ be defined by* (1.2). *Then we obtain for $f\in C(I)$*

$$\|f-K_n^\alpha(f)\|_\infty\le C\frac{1}{n}\left\{\|f\|_\infty+\int_{\frac{n\alpha+1}{n(\alpha+1)}}^{\frac{1}{2}}\frac{\omega_\phi^2(f,t)}{t^2}\frac{dt}{t}\right\}, \tag{1.14}$$

with C a positive constant independent of n and f.

Remark. From (1.14), when $\alpha=0$, we find the analogous result for the classical Kantorovich polynomial.

Proof. It is similar to the proof of Theorem 3.2 in [10]. First we recall that, if $\mathcal{P}_n$ is the best uniform approximation polynomial of degree less than or equal to n to the function f, i.e.,

$$E_n(f)_\infty:=\|f-\mathcal{P}\|_\infty\le C\omega_\phi^2\left(f;\frac{1}{n}\right)_\infty,$$

then [1, Theorem 7.3.1, p. 84]

$$\|\phi^2\mathcal{P}_n''\|_\infty\le C_1 n^2\omega_\phi^2(f;n^{-1})_\infty. \tag{1.15}$$

By the choice of

$$2^{n-1}\le\sqrt{\frac{n(\alpha+1)}{n\alpha+1}}\le 2^n, \tag{1.16}$$

from Lemma 1.5 and (1.15) we obtain

$$\|K_n^\alpha(f)-f\|_\infty\le\|K_n^\alpha(f-\mathcal{P}_{2^n})\|_\infty+\|f-\mathcal{P}_{2^n}\|_\infty+\|K_n^\alpha(\mathcal{P}_{2^n})-\mathcal{P}_{2^n}\|_\infty\le$$

$$\le 2\|f-\mathcal{P}_{2^n}\|_\infty+\frac{C_2}{n}\left(\|\mathcal{P}_{2^n}'\|_\infty+\frac{n\alpha+1}{\alpha+1}\|\phi^2\mathcal{P}_{2^n}''\|_\infty\right)\le$$

$$\le 2E_{2^n}(f)_\infty+\frac{C_3(n\alpha+1)}{n(\alpha+1)}(2^n)^2\,\omega_\phi^2\left(f;\frac{1}{2^n}\right)_\infty+\frac{C_2}{n}\|\mathcal{P}_{2^n}'\|_\infty\le$$

$$\le C^4\left\{\omega_\phi^2\left(f;\frac{1}{2^n}\right)_\infty+\frac{n\alpha+1}{n(\alpha+1)}(2^n)^2\,\omega_\phi^2\left(f;\frac{1}{2^n}\right)_\infty\right\}+$$

$$+\frac{C_2}{n}\left\{\sum_{j=1}^{n}\|\mathcal{P}_{2^j}'-\mathcal{P}_{2^{j-1}}'\|_\infty+\|\mathcal{P}_1'+\mathcal{P}_0'\|_\infty\right\}:=C_4L_1+\frac{C_2}{n}L_2.$$

Now, from Markov-Bernstein inequality, by (1.16) we get for $j=\frac{2i-1}{2}$,

$$L_1 \le 2\omega_\phi^2(f;2^{-n})_\infty \le 2\omega_\phi^2\left(f;\sqrt{\frac{n\alpha+1}{n(\alpha+1)}}\right)_\infty,$$

and

$$L_2 \le C_5\left\{E_0(f)_\infty + \sum_{i=3}^{n} 2^{2i}\omega_\phi^2\left(f;\frac{1}{2^{i-1}}\right)_\infty\right\} \le$$

$$\le C_6\left\{E_0(f)_\infty + \sum_{i=3}^{n}\frac{C_7}{\ln 2}\int_{2^{1-i}}^{2^{2-i}}\frac{\omega_\phi^2(f,t)_\infty}{t^2}\frac{dt}{t}\right\} \le$$

$$\le C_8\left\{E_0(f)_\infty + \int_{\sqrt{\frac{n\alpha+1}{n(\alpha+1)}}}^{\frac{1}{2}}\frac{\omega_\phi^2(f,t)_\infty}{t^2}\frac{dt}{t}\right\}.$$

Finally, we obtain

(1.17)
$$\|K_n^\alpha(f)-f\|_\infty \le$$

$$\le C\left\{\omega_\phi^2\left(f;\sqrt{\frac{n\alpha+1}{n(\alpha+1)}}\right)_\infty + \frac{1}{n}\left[\|f\|_\infty + \int_{\sqrt{\frac{n\alpha+1}{n(\alpha+1)}}}^{\frac{1}{2}}\frac{\omega_\phi^2(f,t)}{t^2}\frac{dt}{t}\right]\right\},$$

with C a constant independent of f and n.

The first term in (1.17) on the right-hand side can be dropped, since

$$\omega_\phi^2\left(f;\sqrt{\frac{n\alpha+1}{n(\alpha+1)}}\right)_\infty \le \frac{C}{n}\omega_\phi^2\left(f;\sqrt{\frac{n\alpha+1}{n(\alpha+1)}}\right)_\infty \int_{\sqrt{\frac{n\alpha+1}{n(\alpha+1)}}}^{\frac{1}{2}}\frac{1}{t^3}dt \le$$

$$\le \frac{C}{n}\int_{\sqrt{\frac{n\alpha+1}{n(\alpha+1)}}}^{\frac{1}{2}}\frac{\omega_\phi^2(f,t)}{t^2}\frac{dt}{t},$$

from which the assertion follows. □

Remark. By the counterexample

$$x\log x - x = f(x)\in C(I),$$

we remark that the integral term in (1.17) cannot be dropped.

APPENDIX ON STANCU OPERATORS

Let S_n^α be the Stancu operator defined by (1.1). We recall that [14]

$$S_n^\alpha(e_i;x)=e_i(x),\quad i=0,1,$$

(1.18)
$$S_n^\alpha(e_2;x)=e_2(x)+\frac{x(1-x)}{n}\frac{1+n\alpha}{1+\alpha}$$

and

$$S_n^\alpha(\Omega_{2,x};x)=\frac{x(1-x)}{n}\frac{1+n\alpha}{1+\alpha}.$$

Then we have

THEOREM A. *Let S_n^α be defined by* (1.1) *and $\phi(x)=\sqrt{x(1-x)}$. Then for $f\in C(I)$*

(1.19)
$$\|f-S_n^\alpha(f)\|_\infty \le 2C\omega_\phi^2\left(f;\sqrt{\frac{1+n\alpha}{n(1+\alpha)}}\right)_\infty,$$

with C a positive constant independent of f and n and $\omega_\phi^2(f)_\infty$ the second modulus of smoothness of Ditzian and Totik.

Proof. Following [1, p. 141], we get, for $\bar{f}\in C^2(I)$, by (1.18)

$$|S_n^\alpha(\bar{f};x)-\bar{f}(x)| = \left|S_n^\alpha\left(\int_x^{\cdot}(t-u)\bar{f}''(u)\,du;x\right)\right| \le$$

$$\le \frac{\|\phi^2\bar{f}''\|_\infty}{\phi^2(x)}S_n^\alpha(\Omega_{2,x};x) \le \frac{1+n\alpha}{n(1+\alpha)}\|\phi^2\bar{f}''\|_\infty.$$

Hence, if we denote by $K_\phi^2(f;t)=\inf_g\{\|f-g\|_\infty + t^2\|\phi^2 g''\|_\infty\}$ the second K-functional [1], then for all $f\in C(I)$ we have

$$|S_n^\alpha(f)-f\|_\infty \le \|S_n^\alpha(f-\bar{f})\|_\infty + \|S_n^\alpha(\bar{f})-\bar{f}\|_\infty + \|f-\bar{f}\|_\infty \le$$

$$\le 2\|f-\bar{f}\|_\infty + \frac{1+n\alpha}{n(1+\alpha)}\|\phi^2\bar{f}''\|_\infty \le 2K_\phi^2\left(f;\frac{1+n\alpha}{n(1+\alpha)}\right)_\infty$$

and from the equivalence between the K-functional and the modulus of smoothness [1], we get (1.19). □

Remark. If $a=0$ in (1.19), then we find a classical result for Bernstein polynomials (see, e.g., [1, p. 3]).

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ON BICRITERIAL TRANSPORT PROBLEMS

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1. INTRODUCTION

Many mathematical programming problems shape themselves as transport problems. In a recent study, it has been shown that more than a half of the applications of linear programming in managing economic processes come to the solving of some transport problems. We shall present a model below.

Tomatoes are cultivated in the farms $A_1, \ldots, A_m$. The daily average production is $a_1, \ldots, a_m$ ware units. The tomatoes are sold in the markets $B_1, \ldots, B_n$. The daily average quantities of tomatoes requested by these markets are $b_1, \ldots, b_n$ ware units. It is known that the price of the transport of a ware unit from the farm A_i $(i \in \{1, \ldots, m\})$ to the market B_j $(j \in \{1, \ldots, n\})$ is c_{ij}. Because tomatoes are perishable, they must be transported as quickly as possible. Let p_{ij} $(i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\})$ be the perishability percentage, per ware unit, of the ware transported from A_i to B_j. It is requested to find out how much ware must be transported from A_i $(i \in \{1, \ldots, m\})$ to B_j $(j \in \{1, \ldots, n\})$, so that:

– all the ware is sold;
– in each market, all the ware that is needed is brought;
– the total cost of the transport is the smallest;
– the quantity of the deteriorated ware is the smallest.

If $x_{ij} \in \mathbb{R}$ $(i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\})$ is the quantity of the ware which will be transported from A_i to B_j, then the model of this problem is

$$v - \min\left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}x_{ij}\right)$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i \in \{1, \ldots, m\}$$

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