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ON BICRITERIAL TRANSPORT PROBLEMS

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1. INTRODUCTION

Many mathematical programming problems shape themselves as transport problems. In a recent study, it has been shown that more than a half of the applications of linear programming in managing economic processes come to the solving of some transport problems. We shall present a model below.

Tomatoes are cultivated in the farms $A_1, \ldots, A_m$. The daily average production is $a_1, \ldots, a_m$ ware units. The tomatoes are sold in the markets $B_1, \ldots, B_n$. The daily average quantities of tomatoes requested by these markets are $b_1, \ldots, b_n$ ware units. It is known that the price of the transport of a ware unit from the farm A_i $(i \in \{1, \ldots, m\})$ to the market B_j $(j \in \{1, \ldots, n\})$ is c_{ij}. Because tomatoes are perishable, they must be transported as quickly as possible. Let p_{ij} $(i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\})$ be the perishability percentage, per ware unit, of the ware transported from A_i to B_j. It is requested to find out how much ware must be transported from A_i $(i \in \{1, \ldots, m\})$ to B_j $(j \in \{1, \ldots, n\})$, so that:

– all the ware is sold;
– in each market, all the ware that is needed is brought;
– the total cost of the transport is the smallest;
– the quantity of the deteriorated ware is the smallest.

If $x_{ij} \in \mathbb{R}$ $(i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\})$ is the quantity of the ware which will be transported from A_i to B_j, then the model of this problem is

$$v - \min\left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}x_{ij}\right)$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i \in \{1, \ldots, m\}$$

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$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j \in \{1,\ldots,n\}$$

$$x_{ij} \geq 0, \quad (i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}.$$

Because in solving problems of the above type we need notions and results that are related with the classic problem, we shall remind them to the readers.

2. PRELIMINARIES

A transport problem (of the cost type) is a linear programming problem of the following type

$$\text{(C)} \qquad \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i \in \{1,\ldots,m\}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j \in \{1,\ldots,n\}$$

$$x_{ij} \geq 0, \quad (i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}.$$

Usually, a transport problem is given in a table of the following type:

c_{11}	...	c_{1n}	a_1
...	...	...	...
c_{m1}	...	c_{mn}	a_m
b_1	...	b_n	

A chain is any system of cells of the type

$$(i_1, j_1),\ (i_1, j_2),\ (i_2, j_2),\ (i_2, j_3), \ldots$$

or

$$(i_1, j_1),\ (i_2, j_1),\ (i_2, j_2),\ (i_3, j_2), \ldots$$

such that any pair of two adjacent cells are situated either in the same row or in the same column and any system formed by three cells from the chain is not situated in the same row or in the same column. If the last cell of the chain is in the same row or column with the first cell, the chain is called cycle.

A transport plane $X = (x_{ij})$ is acyclic if the cells that correspond to $x_{ij} > 0$ do not contain any cycle.

It is known that, if a transport problem admits a transport plane, then it admits at least an acyclic transport plane.

If in an acyclic transport plane $X = (x_{ij})$ the number of elements $x_{ij} > 0$ is $m+n-1$, then the plane is called undegenerated transport plane. If this number is smaller than $m+n-1$, then the transport plane is called degenerated.

If the transport plane $X = (x_{ij})$ is degenerated, we add to the set $\{(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\} : x_{ij} > 0\}$ the elements $(k,h) \in \{1,\ldots,m\} \times \{1,\ldots,n\}$ such that the new set has $m+n-1$ elements and the cells that correspond to it do not form a cycle; this set is called a selection set generated by X and it will be denoted by *X-sel*. Obviously, we can generate more selection sets. The set of the selection sets generated by the plane X will be denoted by *Sel* (X).

The acyclic transport plane $X = (x_{ij})$ is called potential relative to *X-sel*, if there exist real numbers $u_1, \ldots, u_m, v_1, \ldots, v_n$ which satisfy the conditions

$$\text{(1)} \qquad v_j - u_i \leq c_{ij} \quad \text{for all} \quad (i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}$$

and

$$\text{(2)} \qquad v_j - u_i = c_{ij} \quad \text{for all} \quad (i,j) \in X\text{-sel}.$$

If the real numbers $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ satisfy (1) – (2), then the $m+n-$tuple $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ is called potential system relative to *X-sel*.

THEOREM 1.1 (see, for example, [12]). *The transport plane X is an optimal plane if and only if there is a set X-sel ∈ Sel(X) such that X is a potential plane relative to X-sel.*

3. BICRITERIAL TRANSPORT PROBLEMS

In the following, we call a bicriterial transport problem of the cost type, denoted by (BTP), a bicriterial programming problem in which the objective function is a vectorial function $f = (f_1, f_2) : \mathbb{R}^{m \times n} \to \mathbb{R}^2$, given by

$$f_1(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^1 x_{ij}, \quad \text{for all} \quad X = (x_{ij}) \in \mathbb{R}^{m \times n},$$

$$f_2(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^2 x_{ij}, \quad \text{for all} \quad X = (x_{ij}) \in \mathbb{R}^{m \times n},$$

and the constraints are

$$\begin{cases} \sum_{i=1}^{m} x_{ij} = b_j, & j \in \{1, \ldots, n\} \\ \sum_{j=1}^{n} x_{ij} = a_i, & i \in \{1, \ldots, m\} \\ x_{ij} \geq 0, & i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}. \end{cases}$$

The figures of the bicriterial transport problem (BTP) are given in a table of the following type:

$c_{11}^1 \quad c_{11}^2$	$\cdots$	$c_{1n}^1 \quad c_{1n}^2$	a_1
$\cdots$		$\cdots$	
$c_{m1}^1 \quad c_{m1}^2$	$\cdots$	$c_{mn}^1 \quad c_{mn}^2$	a_m
b_1		b_n	

The elements $x_{ij}\ ((i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\})$ will be written in the corresponding cells, under the numbers c_{ij}^1, c_{ij}^2.

The set of the feasible solutions will be denoted by S. Any element $X \in S$ will be called a transport plane.

A transport plane $X \in S$ is called Pareto (or min-efficient) if there is no $Y \in S$ such that

$$f_k(Y) \leq f_k(X), \quad k \in \{1, 2\},$$

at least one of the inequalities being strict.

Because any Pareto transport plane is a Pareto solution of a multicriterial linear programming problem, the properties of the Pareto transport planes set are the same with the properties which we presented in [5] and [6].

Considering the multicriterial transport problem as a multicriterial linear programming problem, for the determination of a Pareto transport plane, we can use any algorithm given in [1]–[3] and [8]–[11]. If, in addition, $x_{ij}\ ((i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\})$ must be an integer, the algorithm given in [7] allows the determination of all the equivalence classes of Pareto transport plane. We can also solve a multicriterial transport problem by using the r-balance points (see [4] and [6]).

The particular form of the multicriterial linear programming problem which corresponds to the multicriterial transport problem allows us to elaborate specific algorithms, as we can see below.

In the following, we shall denote by (T_k) $(k \in \{1, 2\})$ the transport problem

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i \in \{1, \ldots, m\}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j \in \{1, \ldots, n\}$$

$$x_{ij} \geq 0, \quad (i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}.$$

Let $X = (x_{ij})$ be an acyclic transport plane and $X\text{-}sel \in Sel(X)$. For each $k \in \{1, 2\}$, let $(u_1^k, \ldots, u_m^k, v_1^k, \ldots, v_n^k)$ be a solution of the system

$$v_j^k - u_i^k = c_{ij}^k, \quad (i, j) \in X\text{-}sel.$$

We denote by

$$\alpha_{ij}^k = v_j^k - u_i^k - c_{ij}^k,$$

for each $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ and $k \in \{1, 2\}$.

THEOREM 3.1. *Let $k \in \{1, 2\}$. If $X \in S$ is a potential plane of problem (T_k), $X\text{-}sel \in Sel(X)$ and if*

(3) $$\alpha_{ij}^k = v_j^k - u_i^k - c_{ij}^k < 0,$$

for each $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \setminus X\text{-}sel$, then X is a Pareto transport plane for bicriterial transport problem (BTP).

Proof. If X is a potential plane for problem (T_k), then it is an optimum plane for problem (T_k). From (3), it follows that X is the unique optimal transport plane for (T_k). Hence X is a Pareto transport plane for bicriterial transport problem (BTP). ■

THEOREM 3.2. *Let $X = (x_{ij})$ be a potential undegenerated plane for problem (T_1) and let $X\text{-}sel \in Sel(X)$. Let*

$$A_x = \{(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \setminus X\text{-}sel : \alpha_{ij}^1 = 0\}.$$

If there exists $(r,s)\in A_x$ such that $\alpha^2_{rs}>0$, then there is a transport plane Y having the property that

$$f_1(Y)=f_1(X)$$

and

$$f_2(Y)<f_2(X).$$

Proof. We introduce the cell (r,s) in the transport plane X. This will have a cycle C. We travel through this cycle, starting from the cell (r,s) and we denote its cell by + and –, alternatively, starting with the cell (r,s) which gets the + sign. The cells of the cycle denoted by + form a semichain L^+, and the cells of the cycle denoted by – form a semichain L^-. We analyse the elements x_{ij} of the transport plane X situated in the semichain L^- and let

$$\theta=\min\{x_{ij}:(i,j)\in L^-\},$$

which is contained, for example, in cell (u,t). From the elements x_{ij} situated in the semichain L^- we subtract the number θ, and to the elements x_{ij} situated in the semichain L^+ we add the number θ. The other elements, which are not in the cycle C, remain the same. We obtain a new transport plane Y to which we attach the set

$$Y\text{-}sel=X\text{-}sel\cup\{(r,s)\}\setminus\{(u,t)\}.$$

Let us denote by M the set of the cells which are not in the cycle. Then we have

$$f_1(Y)=\sum_{(i,j)\in M}c^1_{ij}y_{ij}+\sum_{(i,j)\in L^+}c^1_{ij}y_{ij}+\sum_{(i,j)\in L^-}c^1_{ij}y_{ij}=$$

$$=\sum_{(i,j)\in M}c^1_{ij}x_{ij}+\sum_{(i,j)\in L^+}c^1_{ij}(x_{ij}+\theta)+\sum_{(i,j)\in L^-}c^1_{ij}(x_{ij}-\theta)=$$

$$=f_1(x)-\theta\cdot\alpha^1_{rs}=f_1(X).$$

Computing $f_2(Y)$ we obtain

$$f_2(Y)=\sum_{(i,j)\in M}c^2_{ij}y_{ij}+\sum_{(i,j)\in L^+}c^2_{ij}y_{ij}+\sum_{(i,j)\in L^-}c^2_{ij}y_{ij}=$$

$$=\sum_{(i,j)\in M}c^2_{ij}x_{ij}+\sum_{(i,j)\in L^+}c^2_{ij}(x_{ij}+\theta)+\sum_{(i,j)\in L^-}c^2_{ij}(x_{ij}-\theta)=$$

$$=f_2(x)-\theta\cdot\alpha^2_{rs}<f_2(X).$$

Hence, transport plane X is not Pareto.

Using Theorems 3.1 and 3.2, we can present the following algorithm for the determination of a Pareto transport plane for bicriterial transport problems.

ALGORITHM

1. Using the potential method, we determine an optimal transport plane $X=(x_{ij})$ for problem (T_1) and we attach to it the set $X\text{-}sel\in Sel(X)$.

2. We determine a solution $(u^1_1,\dots,u^1_m,v^1_1,\dots,v^1_n)$ of the potential system

$$v_j-u_i=c^1_{ij},\quad (i,j)\in X\text{-}sel.$$

3. We compare to zero each of the numbers $\alpha^1_{ij}=v^1_j-u^1_i-c^1_{ij},(i,j)\in\{1,\dots,m\}\times\{1,\dots,n\}\setminus X\text{-}sel.$

a) If $\alpha^1_{ij}<0$, for any $(i,j)\in\{1,\dots,m\}\times\{1,\dots,n\}\setminus X\text{-}sel$, then X is a Pareto transport plane and the algorithm stops.

b) If there exists $(r,s)\in\{1,\dots,m\}\times\{1,\dots,n\}\setminus X\text{-}sel$ such that $\alpha^1_{rs}=0$, then we go to step 4.

4. Put

$$A^1_X=\{(i,j)\in\{1,\dots,m\}\times\{1,\dots,n\}:\alpha^1_{ij}=0\}.$$

5. We solve the system

$$v_j-u_i=c^2_{ij},\quad (i,j)\in X\text{-}sel$$

and let $(u^2_1,\dots,u^2_m,v^2_1,\dots,v^2_n)$ be a solution of it.

6. We compare to zero each of the numbers $\alpha^2_{ij}=v^2_j-u^2_i-c^2_{ij},(i,j)\in A^1_X$.

Put

$$A^2_X=\{(i,j)\in A^1_X:\alpha^2_{ij}>0\}.$$

7. We compare A^2_X to the empty set Ø.

a) If $A^2_X\neq\emptyset$, then the transport plane X is a Pareto transport plane and the algorithm stops.

b) If $A^2_X=\emptyset$, then we go to step 8.

8. We choose a pair $(r,s)\in A^2_X$ such that

$$\alpha^2_{rs}=\max\{\alpha^2_{ij}:(i,j)\in A^1_X\}.$$

9. Put $B = \{(r,s)\} \cup X\text{-}sel$. The set B will have a cycle C. We travel through this cycle, starting from the cell (r, s) and we denote its cells by + and –, alternatively, starting with the cell (r, s) which gets the + sign. The cells of the cycle denoted by + form a semichain L^+ and the cells of the cycle denoted by – form a semichain L^-. We analyse the elements x_{ij} of the transport plane X situated in the semichain L^- and let $\theta = \min\{x_{ij} : (i,j)L^-\}$ which is contained, for example, in cell (u, t). From the elements x_{ij} situated in the semichain L^- we subtract the number θ, and to the elements x_{ij} situated in the semichain L^+ we add the number θ. The other elements, which are not in the cycle C, remain the same. We obtain a new transport plane X to which we attach the set X-sel. We go to step 4.

Example. Let us consider the following numerical example:

4 3	3 6	6 8	6 8	102
2 4	1 2	8 4	4 5	136
4 2	3 4	5 6	9 8	172
151	122	83	54	

Using the potential method, we obtain that problem (T_1) has the optimal solution

$$X = \begin{pmatrix} 48 & 0 & 0 & 54 \\ 103 & 33 & 0 & 0 \\ 0 & 89 & 83 & 0 \end{pmatrix}.$$

We have $X\text{-}sel = \{(1,1), (1,4), (2,1), (2,2), (3,2), (3,3)\}$,

$$(u_1^1, u_2^1, u_3^1, v_1^1, v_2^1, v_3^1, v_4^1) = (0, 2, 0, 4, 3, 5, 6),$$

$$A_X^1 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (3,1), (3,2), (3,3)\},$$

$$(u_1^2, u_2^2, u_3^2, v_1^2, v_2^2, v_3^2, v_4^2) = (0, -1, -3, 3, 1, 3, 8),$$

$$A_X^2 = \{(2,4), (3,1), (3,4)\}.$$

We choose $(r,s) = (2,4)$. We obtain $\theta = 54, (u,t) = (1,4)$, and

$$X = \begin{pmatrix} 102 & 0 & 0 & 0 \\ 49 & 33 & 0 & 54 \\ 0 & 89 & 83 & 0 \end{pmatrix}.$$

We have

$$X\text{-}sel = \{(1,1), (2,1), (2,2), (2,4), (3,2), (3,3)\},$$

$$(u_1^2, u_2^2, u_3^2, v_1^2, v_2^2, v_3^2, v_4^2) = (0, -1, -3, 3, 1, 3, 4),$$

$$A_X^2 = \{(3,1)\}.$$

We choose $(r,s) = (3,1)$. We obtain $\theta = 50, (u,t) = (2,1)$ and

$$X = \begin{pmatrix} 102 & 0 & 0 & 0 \\ 0 & 82 & 0 & 54 \\ 49 & 40 & 83 & 0 \end{pmatrix}.$$

We have $(u_1^2, u_2^2, u_3^2, v_1^2, v_2^2, v_3^2, v_4^2) = (0, 3, 1, 3, 5, 7, 8)$ and $A_X^2 = \emptyset$. Hence

$$X = \begin{pmatrix} 102 & 0 & 0 & 0 \\ 0 & 82 & 0 & 54 \\ 49 & 40 & 83 & 0 \end{pmatrix}$$

is a Pareto transport plane. We have $f(X) = (1437, 1496)$.

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