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IMPROVING THE RATE OF CONVERGENCE OF SOME NEWTON-LIKE METHODS FOR THE SOLUTION OF NONLINEAR EQUATIONS CONTAINING A NONDIFFERENTIABLE TERM

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of a nonlinear equation

(1) $$F(x)+G(x)=0,$$

where F, G are defined on a closed convex subset D of a Banach space E_1 with values in a Banach space E_2. The operator F is Fréchet-differentiable on D whereas G is only continuous there.

We use the Newton-like method given by

(2) $$x_{n+1}=x_n+A_n^{-1}(F(x_n)+G(x_n)) \quad (n\geq 0)$$

to generate a sequence $\{x_n\}$ $(n\geq 0)$ converging to x^*. Here A_n is a linear operator approximating $F'(x_n)$ $(n\geq 0)$. Sufficient conditions for the convergence of (2) to x^* have been given by several authors ([1], [2], [3], [4], [5], [7], [8], [11] and [12]). Recently, Cătinaş in [5] has used (2) for

(3) $$A_n=F'(x_n)+[x_{n-1},x_n;G] \quad (n\geq 1),$$

where $[x, y; G]$ denotes a divided difference of order one of G on D for $x,y\in E_1$. This way Cătinaş has managed to show that the order of convergence denoted by λ lies in $\left[\frac{1+\sqrt{5}}{2},2\right]$. Cătinaş has also showed that iteration (2) is faster than

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iterations appearing in [1], [2], [4], [5], [7], [8], [11] and [12] for choices of A_n other than the one given by (3).

In [3] we used (2) for

$$(4)\quad A_n=[x_n,x_{n-1};F]+[x_{n-2},x_n;F]-[x_{n-2},x_{n-1};F]+[x_{n-1},x_n;G]\quad (n\geq 0).$$

We showed that $\lambda\in\left[\frac{1+\sqrt{5}}{2},1.839...\right]$. Sufficient conditions were also provided under which our error bounds are sharper than all previous ones (in particular those in [5]). We also note that our method is cheaper to use than that in [5].

In this study we have made a further attempt to improve the rate of convergence of iteration (2) by choosing A_n appropriately. Sufficient convergence conditions as well as an error analysis have been provided.

Finally we show that our error bounds are smaller than all earlier ones ([1], [2], [3], [4], [5], [7], [8], [9], [10], [11] and [12]).

2. CONVERGENCE ANALYSIS

We need the following definitions on divided differences [4], [9], [10].

DEFINITION 1. An operator denoted by $[x_0,y_0;H]$ belonging to the space $L(D,E_2)$, $D\subseteq E_1$ (the Banach space of bounded linear operators from E_1 to E_2 is called the first order divided difference of the operator $H:D\to E_2$ at the points $x_0,y_0\in D$ if the following hold:

(5) (a) $[x_0,y_0;H](y_0-x_0)=H(y_0)-H(x_0)$, for $x_0\neq y_0$.

(b) If H is Fréchet-differentiable at $x_0\in D$, then $[x_0,x_0;H]=H'(x_0)$.

DEFINITION 2. An operator denoted by $[x_0,y_0,z_0;H]$ belonging to the space $L(D,L(D,E_2))$ is called the second order divided difference of the operator $H:D\subseteq E_1\to E_2$ at the points $x_0,y_0,z_0\in D$ if the following hold:

(6) (a) $[x_0,y_0,z_0;H](z_0-x_0)=[y_0,z_0;H]-[x_0,y_0;H]$.

(b) If H is twice Fréchet-differentiable at $x_0\in D$, then

$$[x_0,x_0,x_0;H]=\frac{1}{2}H''(x_0).$$

We can prove the following semilocal result concerning the convergence of iteration (2).

THEOREM. *Assume that there exist points* $x_0,x_1\in D$ *and nonnegative real numbers* R, ε *and* m *such that:*

(a) $U(x_1,R)=\{x\in E_1\,|\,\|x-x_1\|\leq R\}\subseteq D$;

(b) *the operators* F, G *have divided differences of order one denoted by* $[x,y;F]$ *and* $[x,y;G]$ *respectively for all* $x,y\in U(x_1,R)$;

(c) *the linear operators* A_n *are invertible for all* $n\geq 0$ *and*

$$\|A_n^{-1}B_n\|\leq p_n\|x_{n-1}-x_{n-2}\|+q_n\|x_n-x_{n-1}\|=\varepsilon_n\quad (n\geq 1),$$

for some nonnegative sequences $\{p_n\},\{q_n\}$ $(n\geq 1)$ *with*

$$p_n+q_n\leq\varepsilon\quad (n\geq 1),$$

where

$$B_n=[x_{n-1},x_n;F]+[x_{n-1},x_n;G]-A_{n-1}\quad (n\geq 1),$$

(d) *the points* x_0, x_1 *satisfy* $\|x_1-x_0\|\leq m$;

(e) *the following conditions hold:*

$$(10)\qquad \|x_2-x_1\|\leq\|x_1-x_0\|,$$

with x_2 *given by* (2) *for* $n=1$,

$$(11)\qquad r=m\varepsilon<1,$$

$$(12)\qquad R\geq\frac{m}{r}\sum_{k=1}^{\infty}r_k,$$

$$(13)\qquad r_k=r^{s_k},\ (k\geq 0),$$

where $\{s_k\}$ *is the Fibonacci's sequence*

$$(14)\qquad s_0=s_1=1,\ s_{k+1}=s_k+s_{k-1},\quad (k\geq 0).$$

Then:

(i) *the sequence* $\{x_n\}$ $(n\geq 0)$, *generated by* (2) *is well defined, remains in* $U(x_0,R)$ *and converges to a solution* $x^*\in U(x_0,R)$ *of the equation* $F(x)+G(x)=0$;

(ii) *the following a priori error estimates hold*

$$(15)\qquad \|x^*-x_n\|\leq\frac{mt_n}{r(1-t_n^{\ell-1})},\quad (n\geq 1),$$

where

$$(16)\qquad t_n=r^{\frac{\ell^n}{\sqrt{5}}},\quad (n\geq 1)\quad and\quad \ell=\frac{1+\sqrt{5}}{2};$$

(iii) *moreover, if there exists a nonnegative number g such that*

(17) $$\|A_n^{-1}B_n^*\| \le g < 1, \quad (n \ge 1),$$

where

(18) $$B_n^* = [y^*, x_n; F] + [y^*, x_n; G] - A_n, \quad (n \ge 1),$$

with y^* *satisfying*

$$F(y^*) + G(y^*) = 0 \quad \textit{and} \quad y^* \in U(x_0, R),$$

then

$$x^* = y^*.$$

Proof. We shall show by induction that, for all $n \ge 2$

(19) $$x_n \in U(x_1, R),$$

(20) $$\|x_n - x_{n-1}\| \le \|x_{n-1} - x_{n-2}\|$$

and

(21) $$\|x_n - x_{n-1}\| \le \frac{m}{r} r_{n-1}.$$

For $n = 2$ relations (19)–(21) follow from hypotheses (d) and (e). Suppose relations (19)–(21) hold for $n = 2, 3, \ldots, k$, where $k \ge 2$. Since $x_k, x_{k-1} \in U(x_1, R)$ and A_k is invertible, via (2) we can compute x_{k+1}. Using (2), we can obtain the approximation

(22) $$F(x_k) + G(x_k) = F(x_k) + G(x_k) - F(x_{k-1}) - G(x_{k-1}) - A_{k-1}(x_k - x_{k-1}) =$$
$$= ([x_{k-1}, x_k; F] + [x_{k-1}, x_k; G] - A_{k-1})(x_k - x_{k-1}) =$$
$$= B_k(x_k - x_{k-1}) \quad \text{(by (9))}.$$

By (2), (7), (8) and (22) we get

(23) $$\|x_{k+1} - x_k\| \le \varepsilon_k \|x_k - x_{k-1}\| \le \varepsilon \|x_{k-1} - x_{k-2}\| \cdot \|x_k - x_{k-1}\|.$$

From the induction hypotheses, (23) gives on the one hand, that

$$\|x_{k+1} - x_k\| \le \frac{\varepsilon}{r} r_{k-2} m \|x_k - x_{k-1}\| = r_{k-2}\|x_k - x_{k-1}\| < \|x_k - x_{k-1}\|,$$

that is, (20) for $n = k + 1$, and, on the other hand

$$\|x_{k+1} - x_k\| \le r_{k-2}\|x_k - x_{k-1}\| \le r_{k-2} r_{k-1} \frac{m}{r} = \frac{m}{r} r_k,$$

which shows (21) for $n = k + 1$.

We must also show that $x_{k+1} \in U(x_1, R)$. Indeed, from the induction hypotheses and the triangle inequality we get

$$\|x_{k+1} - x_1\| \le \|x_2 - x_1\| + \|x_3 - x_2\| + \ldots + \|x_{k+1} - x_k\| \le \frac{m}{r} \sum_{j=1}^{k} r_k < R.$$

We must show that sequence $\{x_n\}\,(n \ge 0)$ is Cauchy. We have that the Fibonacci's sequence $\{s_k\}\,(k \ge 0)$ given by (14) can also be written as

$$s_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \ge \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k} = \frac{\ell^k}{\sqrt{5}} \quad (k \ge 1).$$

Therefore, for any $k \ge 1, j \ge 1$ we get

$$\|x_{k+j} - x_k\| \le \|x_{k+1} - x_k\| + \|x_{k+2} - x_{k+1}\| + \ldots + \|x_{k+j} - x_{k+j-1}\| \le$$
$$\le \frac{m}{r} \sum_{i=k}^{k+j-1} r_i \le \frac{m}{r} \sum_{i=k}^{k+j-1} r^{\frac{\ell^i}{\sqrt{5}}}.$$

Moreover, by Bernoulli's inequality we get

(24) $$\|x_{k+j} - x_k\| \le \frac{m}{r} r^{\frac{\ell^k}{5}} \left[1 + r^{\frac{\ell^{k+1} - \ell^k}{\sqrt{5}}} + r^{\frac{\ell^{k+2} - \ell^k}{\sqrt{5}}} + \ldots + r^{\frac{\ell^{k+j-1} - \ell^k}{\sqrt{5}}}\right] \le$$
$$\le \frac{m}{r} r^{\frac{\ell^k}{\sqrt{5}}} \left[1 + r^{\frac{\ell^k(\ell-1)}{\sqrt{5}}} + r^{\frac{\ell^k[1+2(\ell-1)-1]}{\sqrt{5}}} + \ldots + r^{\frac{\ell^k[1+(j-1)(\ell-1)-1]}{\sqrt{5}}}\right] =$$
$$= \frac{m}{r} r^{\frac{\ell^k}{\sqrt{5}}} \frac{1 - r^{\frac{\ell^k(\ell-1)}{\sqrt{5}} j}}{1 - r^{\frac{\ell^k(\ell-1)}{\sqrt{5}}}} \quad (k \ge 1).$$

By (11) and (24) it follows that the sequence $\{x_n\}\,(n \ge 0)$ is Cauchy in a Banach space E_1, and so it converges to some point $x^* \in U(x_1, R)$ (since $U(x_1, R)$ is a closed set). By letting $n \to \infty$ in (2), we obtain $F(x^*) + G(x^*) = 0$; that is, $x^* \in U(x_1, R)$ is a solution of equation (1). Moreover, by letting $j \to \infty$ in (24) we obtain (15).

Furthermore, to show that x^* is the unique solution of equation (1) in $U(x_1, R)$, let us assume that $y^* \in U(x_1, R)$ is a solution of equation (1) too.

Using the approximation

(25) $$x_{k+1} - y^* = x_k - y^* - A_k^{-1}(F(x_k) + G(x_k)) =$$

$$= -A_k^{-1}[F(x_k) - F(y^*) + G(x_k) - G(y^*) - A_k(x_k - y^*)] =$$

$$= -A_k^{-1} B_k^* (x_k - y^*).$$

and hypothesis (17), we get

(26) $$\|x_{k+1} - y^*\| \le \|A_k^{-1} B_k^*\| \|x_k - y^*\| \le$$

$$\le g\|x_k - y^*\| \le \ldots \le g^k \|x_1 - y^*\| \le g^k R.$$

Since $0 \le g < 1$, by letting $k \to \infty$ in (26) we get $\lim_{k\to\infty} x_k = y^*$. But we have also showed that $\lim_{k\to\infty} x_k = x^*$. Hence, we deduce $x^* = y^*$.

That completes the proof of the Theorem.

Remark 1. Let us consider some special choices for the linear operators A_n. Set

(27) $$A_n = [x_n, y_n; F] + [h_n, z_n; F] - [h_n, z_{n-1}; F] + [v_n, x_n; G] \quad (n \ge 0),$$

where the sequences $\{y_n\}, \{z_n\}, \{v_n\}, \{h_n\} \in U(x_1, R)$ $(n \ge 0)$ are given by

$$y_n = x_n + \alpha_n(x_{n-1} - x_n), \quad z_n = z_{n-1} + \beta_n(x_{n-1} - x_n), \quad z_{-1} = x_{-1} \in U(x_1, R),$$

$$v_n = x_n + \gamma_n(x_{n-1} - x_n) \quad (n \ge 0),$$

for some linear operator sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ $(n \ge 0)$ with $\gamma_n \ne 0$ $(n \ge 0)$. Assume that there exist nonnegative numbers a, b, c and a real sequence $\{a_n\}$ $(n \ge 0)$ such that for all $x, y, v, w, z \in U(x_1, R)$

(28) $$\|A_0^{-1}([x, y; F] - [v, w; F])\| \le b(\|x - v\| + \|y - w\|),$$

(29) $$\|A_n^{-1} A_0\| \le a_n \le a \quad (n \ge 0),$$

and

(30) $$\|A_n^{-1}[x, y, z; G]\| \le c,$$

where $[x, y, z; G]$ is the divided difference of order two of G on $U(x_1, R)$. Then from the approximation

$$G(x_n) - G(x_{n-1}) - [v_{n-1}, x_{n-1}; G](x_n - x_{n-1}) =$$

$$= ([x_{n-1}, x_n; G] - [v_{n-1}, x_{n-1}; G])(x_n - x_{n-1}) =$$

$$= [v_{n-1}, x_{n-1}, x_n; G](x_n - v_{n-1})(x_n - x_{n-1}),$$

hypotheses (28), (29) and (30) we get

(31) $$\|A_0^{-1}(G(x_n) - G(x_{n-1}) - [v_{n-1}, x_{n-1}; G](x_n - x_{n-1}))\| \le$$

$$\le c\|x_n - v_{n-1}\| \|x_n - x_{n-1}\| \le$$

$$\le c(\|x_n - v_{n-1}\| + \|\gamma_n\| \|x_{n-1} - x_{n-2}\|)\|x_n - x_{n-1}\| \quad (n \ge 1).$$

The sequence $\{a_n\}$ $(n \ge 1)$ can be computed as follows. Let us assume that there exist $\bar{c} \ge 0$ such that

(32) $$\|A_0^{-1}([x, y; G] - [v_0, x_0; G])\| \le \bar{c}(\|x - v_0\| + \|y - x_0\|),$$

for all $x, y, v_0, x_0 \in U(x_1, R)$. Then from the approximation

$$A_0^{-1}(A_n - A_0) = A_0^{-1}\{[x_n, y_n; F] + [h_n, z_n; G] - [h_n, z_{n-1}; G] +$$

$$+ [v_n, x_n; G] - [x_0, y_0; F] - [h_0, z_0; F] + [h_0, z_{-1}; F] - [v_0, x_0; G]\},$$

we can get as before

$$\|A_0^{-1}(A_n - A_0)\| \le \bar{a}_n \quad (n \ge 1),$$

where

$$\bar{a}_n = b(\|x_n - x_0\| + \|y_n - y_0\| + 2\|h_n - h_0\| + \|z_n - z_0\| + \|z_{n-1} - z_{-1}\|) +$$

$$+ \bar{c}(\|v_n - x_0\| + \|x_n - x_0\|) \quad (n \ge 1),$$

and $\bar{a}_n < 1$ if the function $a(r) = (11b + 2\bar{c})r + (b + 2\bar{c})m$ satisfies

(33) $$a(R) < 1,$$

since

$$\bar{a}_n \le a(R) \quad (n \ge 1).$$

It follows from the Banach lemma on invertible operators that A_n^{-1} exists $(n \ge 1)$ and

$$\|A_n^{-1} A_0\| \le (1 - \bar{a}_n)^{-1}.$$

We can now set $a_n = (1 - \bar{a}_n)^{-1}$ $(n \ge 1)$ and $a = (1 - a(R))^{-1}$. Moreover, from the approximation

(34) $$F(x_n) - F(x_{n-1}) -$$

$$-([x_{n-1}, y_{n-1}; F] + [h_{n-1}, z_{n-1}; F] - [h_{n-1}, z_{n-1}; F])(x_n - x_{n-1}) =$$

$$= ([x_{n-1}, x_n; F] - [x_{n-1}, y_{n-1}; F] - [x_{n-1}, z_{n-1}; F]) + [x_{n-1}, z_{n-2}; F])(x_n - x_{n-1}),$$

hypotheses (28), (29), (30) and (32) we also get

(35) $$\|A_0^{-1}\{F(x_n)-F(x_{n-1})-$$

$$-([x_{n-1},y_{n-1};F]+[h_{n-1},z_{n-1};F]-[h_{n-1},z_{n-2};F])(x_n-x_{n-1})\}\|\leq$$

$$\leq b(\|x_n-y_{n-1}\|+\|z_{n-1}-z_{n-2}\|)\,\|x_n-x_{n-1}\|\leq$$

$$\leq b(\|x_n-x_{n-1}\|+\|\alpha_{n-1}+\beta_{n-1}\|\,\|x_{n-2}-x_{n-1}\|)\,\|x_n-x_{n-1}\| \quad (n\geq 1).$$

Define the sequences $\{d_n\},\{\delta_n\}\,(n\geq 1)$ by

(36) $$d_n=(b+c)\,a_n \quad\text{and}\quad \delta_n=a_n(c\|\gamma_n\|+b(\|\alpha_n\|+\|\beta_n\|)] \quad (n\geq 0).$$

Then from (2), (35) and (36) we get for $n\geq 1$

(37) $$\|x_{n+1}-x_n\|\leq(d_n\|x_n-x_{n-1}\|+\delta_n\|x_{n-1}-x_{n-2}\|)\,\|x_n-x_{n-1}\| \quad (n\geq 1).$$

Hence we can set

(38) $$p_n=\delta_n \quad\text{and}\quad d_n=q_n \quad (n\geq 0).$$

We can impose additional conditions on the sequences $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ $(n\geq 0)$ that will guarantee that $\{y_n\},\{z_n\},\{v_n\}\in U(x_1,R)$. Let us assume that there exist nonnegative numbers α, β, and γ such that

$$\|\alpha_n\|\leq\alpha,\ \|\beta_n\|\leq\beta \quad\text{and}\quad \|\gamma_n\|\leq\gamma \quad (n\geq 0).$$

Then, from the approximations

$$y_n-x_1=(x_n-x_1)+\alpha_n(x_{n-1}-x_n)$$

$$v_n-x_1=(x_n-x_1)+\gamma_n(x_{n-1}-x_n)$$

$$z_n-x_1=(z_{n-1}-x_1)+\beta_n(x_{n-1}-x_n),$$

we can have

(39) $$\|x_n-x_1\|+\|\alpha_n\|\,\|x_{n-1}-x_n\|\leq\frac{m}{r}\sum_{i=1}^{n-1}r_i+\frac{m\alpha}{r}r_{n-1},$$

(40) $$\|v_n-x_1\|+\|\gamma_n\|\,\|x_{n-1}-x_n\|\leq\frac{m}{r}\sum_{i=1}^{n-1}r_i+\frac{m\gamma}{r}r_{n-1},$$

and

(41) $$\|z_n-x_1\|\leq\|z_{-1}-x_1\|+\beta\sum_{i=0}^{n-1}\|x_i-x_{i-1}\|\leq\|z_{-1}-x_1\|+\frac{\beta m}{r}\sum_{i=0}^{n-1}r_i.$$

Hence $y_n,v_n,z_n\in U(x_1,R)\,(n\geq 0)$ if the right hand sides of the last three inequalities are respectively bounded above by R.

Finally, the uniqueness of the solution x^* can be extended in the ball $U(x_1,R_1)$ for $R_1\geq R$ provided that the following inequality holds

(42) $$g=(5(b+\bar{c})\,R+(b+c)\,R_1+2m\bar{c})(1-a(R))^{-1}<1.$$

Indeed, as in (25) we get

$$B_n^*=([y^*,x_n;F]-[x_n,y_n;F])+([y^*,x_n;G]-[x_0,v_0;G])+$$

$$+([x_0,v_0;G]-[v_n,x_n;G])+([h_n,z_{n-1};F]-[h_n,z_n;F]).$$

Composing both sides of the above approximation by A_0^{-1}, we easily deduce that $A_0^{-1}B_k^*$ (in norm) is bounded above by the expression in the bracket of inequality (42). Hence, as in the proof of the Theorem, we deduce $x^*=y^*$.

Concluding, we note that we have showed: if hypotheses (c) of the Theorem are replaced by (27), (28), (29), (30), (32) and (42), then the conclusions of the Theorem hold in the ball $U(x_1,R_1)$.

Remark 2. Iteration (2) reduces to (4) considered in [5] if the linear operators $\{A_n\}\,(n\geq 0)$ are given by (27) for $\alpha_n=0,\gamma_n=I,\beta_n=0,z_n=0\,(n\geq 0)$.

Using the notation introduced in [5], we can set

(43) $$\varepsilon_n^1=MK\|x_{n-1}-x_{n-2}\|+M\left(\frac{\ell}{2}+K\right)\|x_n-x_{n-1}\| \quad (n\geq 2).$$

Hence our error bounds (15) will be smaller than those in [5], say if (see also (38))

(44) $$p_n\leq MK \quad\text{and}\quad q_n\leq M\left(\frac{\ell}{2}+K\right) \quad (n\geq 2)$$

and our initial error bounds $\|x_1-x_0\|$ are not greater than those in [5]. The choice of p_n,q_n given by (38) shows that conditions (44) will be true if a_n,α_n,β_n and $\gamma_n\,(n\geq 0)$ are "small" enough.

Remark 3. Moreover, iteration (2) reduces to (1) considered in [11] if the linear operators $\{A_n\}\,(n\geq 0)$ are given by (27) for $G=0,\alpha_n=I,\beta_n=I,z_n=x_n$ and $h_n=x_{n-2}\,(n\geq 0)$. Using the notation introduced in [11] we can set

(45) $$\varepsilon_n^2=q_0\|x_{n-3}-x_{n-1}\|\,\|x_{n-2}-x_{n-1}\|+p_0\|x_n-x_{n-1}\| \quad (n\geq 1).$$

Hence our error bounds (15) will be smaller in this case, say if

(46) $$p_n \le \|x_{n-3} - x_{n-1}\| \text{ and } q_n \le p_0 \quad (n \ge 1).$$

Observations similar to those made at the end of Remark 2 can now follow.

Remark 4. Furthermore, iteration (2) reduces to (5) considered in [3] if the linear operators $\{A_n\}$ $(n \ge 0)$ are given by (27) for

$$\alpha_n = I,\ \beta_n = 0,\ \gamma_n = I,\quad z_n = x_n,\ h_n = x_{n-2} \quad (n \ge 0).$$

Using the notation introduced in [3], we can set

(47) $$\varepsilon_n^3 = (c_4 + c_2\|x_{n-3} - x_{n-1}\|)\|x_{n-1} - x_{n-2}\| + (c_1 + c_4)\|x_n - x_{n-1}\| \quad (n \ge 1).$$

Hence our bounds will be smaller in this case say if

(48) $$p_n \le c_4 + c_2\|x_{n-3} - x_{n-1}\| \text{ and } q_n \le c_1 + c_4 \quad (n \ge 1).$$

Remark 5. Our results extend to include perturbed Newton-like methods of the form

(49) $$x_{n+1} = x_n - A_n^{-1}(F(x_n) + G(x_n)) - w_n \quad (n \ge 0).$$

The points $\{w_n\}$ $(n \ge 0)$ are determined in such a way that iteration $\{x_n\}$ $(n \ge 0)$ converges to a solution x^* of equation (1). The importance of studying perturbed Newton-like methods comes from the fact that many commonly used variants of Newton's method can be considered procedures of this type. Indeed, approximation (49) characterizes any iterative process in which corrections are taken as approximate solutions of Newton equations. We also note that if, for example, an equation on the real line is solved, $F(x_n) > 0$ $(n \ge 0)$, and $A(x_n)$ $(n \ge 0)$ overestimates the derivative $x_n - A_n^{-1}F(x_n)$, is always larger than the corresponding Newton iterate. In such cases, a positive w_n $(n \ge 0)$ correction term is appropriate. Let us assume that there exists a real sequence $\{u_n\}$ $(n \ge 0)$ such that

(50) $$\|A_n(w_n) - A_{n-1}(w_{n-1})\| \le u_n \quad (n \ge 1).$$

Moreover, there exist real sequences $\{\ell_n\}$, $\{m_n\}$ $(n \ge 0)$ such that

(51) $$u_n \le (\ell_n\|x_{n-1} - x_{n-2}\| + m_n\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\| \quad (n \ge 1).$$

Set

$$\overline{p}_n = p_n + \ell_n \text{ and } \overline{q}_n = q_n + m_n \quad (n \ge 1).$$

Furthermore, assume that sequence $\{w_n\}$ $(n \ge 0)$ is null. Finally, assume that the rest of the hypotheses of the Theorem are true with $\overline{p}_n, \overline{q}_n$ replacing p_n, q_n $(n \ge 1)$, respectively. Then it can easily be seen that the conclusions of the Theorem will hold for the perturbed Newton-like method generated by (49). Indeed, for example, approximation (22) will read

$$F(x_k) + G(x_k) + A_k(w_k) = [B_k + (A_k(w_k) - A_{k-1}(w_{k-1}))](x_k - x_{k-1}) \quad (n \ge 1)$$

and by using the proof of the theorem, (50) and (51) we can arrive at (23). The rest is left to the motivated reader.

Remark 6. The selection of the points $\{y_n\}, \{z_n\}, \{v_n\}$ $(n \ge 0)$ can be generalized to include a wider range of problems. Let $T_1, T_2, T_3 : D \subseteq E_1 \to E_2$ be given operators. Define for all $n \ge 0$ $y_n = T_1(x_n)$, $z_n - z_{n-1} = T_2(x_n)$, $z_{-1} = x_{-1} \in U(x_1, R) \subseteq D$ and $v_n = T_3(x_n)$. For this choice of T_1, T_2 and T_3 iteration (2) becomes a Steffensen-like method ([8], [9] and [10]). Moreover, operators T_1, T_2 and T_3 must be chosen so that estimate (7) be true. See how this is done, for example, in Remark 1.

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