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THE APPROXIMATION BY SPLINE FUNCTIONS OF THE SOLUTION OF A SINGULARLY PERTURBED BILOCAL PROBLEM

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The singularly perturbed bilocal problems admit exact solutions having both slowly and rapidly varying parts. There are thin transition layers where the solution can jump abruptly, having as effect strong oscillations of the approximate solutions obtained by the method of centered differences, spectral methods, etc.

We define a class of spline functions of degree 5 which are appropriate for these problems and obtain sufficiently good approximate solutions, with attenuated oscillations on the subintervals where the exact solution jumps rapidly.

Let $n \geq 3$, $n \in \mathbb{N}$, and let

(1) $$\Delta_n : -\infty = t_{-1} < a = t_0 < t_1 < \ldots < t_n = b < t_{n+1} = +\infty$$

a division of the real axis.

Denote by $\mathscr{S}_5(\Delta_n)$ the set of functions $s : \mathbb{R} \to \mathbb{R}$ verifying the conditions:

1° $s \in C^4(\mathbb{R})$;

2° $s|_{I_k} \in \mathscr{P}_5$, $I_k = [t_{k-1}, t_k)$, $k = 1, 2, \ldots, n$;

3° $s|_{I_0} \in \mathscr{P}_3, s|_{I_{n+1}} \in \mathscr{P}_3$, $I_0 = [t_{-1}, t_0)$, $I_{n+1} = [t_n, t_{n+1})$,

where $\mathscr{P}_m$ denotes the set of polynomials of degree m.

As concerns the behavior of functions in this class, one can prove

THEOREM 1. *Every function $s \in \mathscr{S}_5(\Delta_n)$ can be written in the form*

$$s(t) = \sum_{i=0}^{3} A_i t^i + \sum_{k=0}^{n} a_k (t - t_k)_+^5, \quad t \in \mathbb{R},$$

where

(3) $$\sum_{k=0}^{n} a_k = 0, \sum_{k=0}^{n} a_k t_k = 0,$$

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and

$$(t-t_k)_+ = \begin{cases} 0 & \text{if } t < t_k, \\ t-t_k & \text{if } t \geq t_k, \end{cases} \quad k = 0,1,\ldots,n.$$

Proof. Let $s \in \mathscr{S}_5(\Delta_n)$. By definition $s^{(4)}(t) = 0$ for all $t \geq b$ so that $s^{(4)}(t) = \sum_{k=0}^{n} a_k (t-t_k) = 0$, for all $t \geq b$, showing that $\sum_{k=0}^{n} a_k = 0$ and $\sum_{k=0}^{n} a_k t_k = 0$. □

THEOREM 2. *Let $f:\mathbb{R}\to\mathbb{R}$ be such that*

$$f(a)=\alpha,\ f(b)=\beta,\ f''(t_k)=\lambda_k,\ k=0,1,2,\ldots,n, \tag{4}$$

where t_k ($t_0 = 1, t_n = b$), $k=\overline{0,n}$, are the knots of the division Δ_n and α, β, λ_k, $k=\overline{0,n}$ are given numbers.

Then there exists a unique function $s_f \in \mathscr{S}_5(\Delta_n)$ such that

$$s_f(a)=\alpha,\ s_f(b)=\beta,\ s_f''(t_k)=\lambda_k,\ k=\overline{0,n}. \tag{5}$$

Proof. Using the representation (2) and imposing the conditions (5), one obtains the system:

$$\begin{aligned} &A_0 + A_1 a + A_2 a^2 + A_3 a^3 = \alpha \\ &A_0 + A_1 b + A_2 b^2 + A_3 b^3 + \sum_{k=0}^{n-1} a_k (b-t_k)^5 = \beta \\ &2A_2 + 6A_3 t_j + 20\cdot\sum_{k=0}^{n} a_k (t_j - t_k)_+^3 = \lambda_j,\quad j=\overline{0,n} \\ &\sum_{k=0}^{n} a_k = 0 \\ &\sum_{k=0}^{n} a_k t_k = 0 \end{aligned} \tag{6}$$

of $n+5$ equations with $n+5$ unknowns $A_0, A_1, A_2, A_3, a_0, a_1, \ldots a_n$.

The system (6) has a unique solution if and only if the corresponding homogeneous system (obtained for $\alpha=\beta=0$, $\lambda_j=0$, $j=\overline{0,n}$) has only the null solution.

If $s \in \mathscr{S}_5(\Delta_n)$ verifies the homogeneous conditions (5) (i.e., with $\alpha=\beta=0$, $\lambda_k = 0$, $j=\overline{0,n}$), then

$$\int_a^b [s^{(4)}(t)]^2\,dt = \int_a^b s^{(4)}(t)\,(s'''(t))'\,dt =$$

$$s^{(4)}(t)\cdot s'''(t)\big|_a^b - \int_a^b s'''(t)\cdot s^{(5)}(t)\cdot dt =$$

$$= -\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} s^{(5)}(t)\cdot s'''(t)\,dt = -\sum_{k=1}^{n} c_k \int_{t_{k-1}}^{t_k} s'''(t)\,dt =$$

$$= -\sum_{k=1}^{n} c_k [s''(t_k) - s''(t_{k-1})] = 0,$$

where $c_k = s^{(5)}(t)|_{I_k}$, $k=\overline{1,n}$.

It follows that $s^{(4)}(t)=0$ for all $t\in[a,b]$. Since the restrictions of s to the intervals I_0, I_{n+1} are in $\mathscr{P}_3$ and $s \in C^4(\mathbb{R})$, it results that $s^{(4)}(t)=0$ for all $t\in\mathbb{R}$. Therefore $s''\in\mathscr{P}_1$ and, taking into account the equalities $s''(t_k)=0$, $k=\overline{0,n}$, $n\geq 3$ (verified by hypothesis), one obtains $s''(t)=0$ for all $t\in\mathbb{R}$. Since $s(a)=s(b)=0$, it follows $s(t)=0$ for all $t\in\mathbb{R}$, which is equivalent to $A_0=A_1=A_2=A_3=0$ and $a_k=0$, $k=\overline{0,n}$. □

COROLLARY 3. a) *There exists a system $\mathscr{B}=\{s_0, s_1, S_0, S_1, \ldots, S_n\}$ of functions in $\mathscr{S}_5(\Delta_n)$ verifying the conditions:*

$$\begin{aligned} &s_0(a)=1,\ s_0(b)=0,\ s_0''(t_k)=0,\ k=\overline{0,n} \\ &s_1(a)=0,\ s_1(b)=1,\ s_1''(t_k)=0,\ k=\overline{0,n} \\ &S_k(a)=0=S_k(b),\ k=\overline{0,n};\ S_k''(t_j)=\delta_{kj},\ k,j=\overline{0,n}. \end{aligned} \tag{7}$$

b) *If $f:\mathbb{R}\to\mathbb{R}$ verifies the conditions* (4) *and $s_f \in \mathscr{S}_5(\Delta_n)$ verifies the conditions* (5) *then*

$$s_f(t) = s_0(t)\cdot f(a) + s_1(t)\cdot f(b) + \sum_{k=0}^{n} S_k(t)\cdot f''(t_k),\ t\in\mathbb{R}. \tag{8}$$

Remark 1. By Corollary 3, $\mathscr{S}_5(\Delta_n)$ is a real linear space of dimension $n+3$, and the system $\mathscr{B}$ is a basis in $\mathscr{S}_5(\Delta_n)$.

Let us introduce now the notations:

(9) $$W_2^4(\Delta_n) := \left\{ \begin{array}{l} g:[a,b]\to \mathbb{R},\ \text{abs.cont.on}\ \ I_k,\ k=\overline{1,n} \\ \qquad \text{and}\ \ g^{(4)} \in L_2[a,b] \end{array} \right\},$$

(10) $$W_{2,f}^4(\Delta_n) := \{ g \in W_2^4(\Delta_n) : g''(t_k) = f''(t_k),\ k=\overline{0,n}\},$$

(11) $$W_{2,f,D}^4(\Delta_n) := \{ g \in W_{2,f}^4(\Delta_n) : g(t_a) = f(a),\ g(b) = f(b)\}.$$

THEOREM 4. If $s \in \mathscr{S}_5(\Delta_n) \cap W_{2,f,D}^4(\Delta_n)$ and $f \in W_2^4(\Delta_n)$, then

(12) a) $\|s_f^{(4)}\|_2 \le \|g^{(4)}\|_2$ *for all* $g \in W_{2,f,D}^4(\Delta_n)$;

(13) b) $\|s_f^{(4)} - f^{(4)}\|_2 \le \|s^{(4)} - f^{(4)}\|_2$ *for all* $s \in \mathscr{S}_5(\Delta_n)$.

Proof. a) We have

$$0 \le \|g^{(4)} - s^{(4)}\|_2^2 = \int_a^b [g^{(4)}(t) - s^{(4)}(t)]^2\, dt =$$

$$= \int_a^b [g^{(4)}(t)]^2\, dt - \int_a^b [s^{(4)}(t)]^2\, dt - 2\int_a^b s^{(4)}(t)[g^{(4)}(t) - s^{(4)}(t)]\, dt.$$

But

$$\int_a^b s^{(4)}(t)[g^{(4)}(t) - s^{(4)}(t)]\, dt =$$

$$= s^{(4)}(t)[g'''(t) - s'''(t)]\big|_a^b - \int_a^b s^{(5)}(t)[g'''(t) - s'''(t)]\, dt =$$

$$= -\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} s^{(5)}(t)[g'''(t) - s'''(t)] \cdot dt =$$

$$= -\sum_{k=1}^{n} c_k \int_{t_{k-1}}^{t_k} [g'''(t) - s'''(t)] \cdot dt = \left(-\sum_{k=1}^{n} c_k \right) \cdot 0 = 0,$$

where $c_k = s^{(5)}|_{I_k}$, $k = 1, 2, \ldots, n$.

Therefore

$$0 \le \|g^{(4)}\|_2^2 - \|s^{(4)}\|_2^2$$

showing that (12) holds.

b) Taking into account the identity

(14) $$\|s^{(4)} - f^{(4)}\|_2^2 = \int_a^b [s^{(4)}(t) - s_f^{(4)}(t)]^2\, dt + \int_a^b [s_f^{(4)}(t) - f^{(4)}(t)]^2\, dt +$$

$$+ 2 \cdot \int_a^b [s^{(4)}(t) - s_f^{(4)}(t)]^2 \cdot [s_f^{(4)}(t) - f^{(4)}(t)]^2\, dt,$$

the inequality (13) will be a consequence of the equality

$$T = \int_a^b [s^{(4)}(t) - s_f^{(4)}(t)][s_f^{(4)}(t) - f^{(4)}(t)]\, dt = 0.$$

Integrating by parts, we get

$$T = [s^{(4)}(t) - s_f^{(4)}(t)][s_f'''(t) - f'''(t)]\big|_a^b -$$

$$- \int_a^b [s^{(5)}(t) - s_f^{(5)}(t)] \cdot [s_f'''(t) - f'''(t)]\, dt =$$

$$= -\sum_{k=1}^{n} C_k(s)\left([s_f''(t_k) - f''(t_k)] - [s_f''(t_{k-1}) - f''(t_{k-1})]\right) = 0.$$

Therefore

(15) $$\|s^{(4)} - f^{(4)}\|_2^2 = \|s^{(4)} - s_f^{(4)}\|_2^2 + \|s_f^{(4)} - f^{(4)}\|_2^2$$

implying

$$\|s_f^{(4)} - f^{(4)}\|_2 \le \|s^{(4)} - f^{(4)}\|_2 . \quad \square$$

COROLLARY 5. *If* $f \in W_2^4(\Delta_n)$ *and* $s_f \in \mathscr{S}_5(\Delta_n)$ *is given by* (8), *then the following relations hold:*

(16) $$\|f^{(4)}\|_2^2 = \|s_f^{(4)}\|_2^2 + \|f^{(4)} - s_f^{(4)}\|_2^2,$$

(17) $$\|s_f^{(4)}\|_2 \le \|f^{(4)}\|_2,$$

(18) $$\|f^{(4)} - s_f^{(4)}\|_2 \le \|f^{(4)}\|_2 .$$

Proof. Since (15) holds for every $s \in \mathscr{S}_5(\Delta_n)$, one obtains (16) by taxing $s \equiv 0$ in (15).

The inequalities (17) and (18) are immediate consequences of the equality (16).

Remark 2. 1° The property expressed by the inequality (12) is called the minimum norm property.

2° The property exprimed by the inequality (13) is called the best approximation property.

APPLICATION

Consider the singularly perturbed bilocal problem

(D) $$\varepsilon y'' = f(t, y, y'),\ t \in [a,b],\ \varepsilon > 0$$
$$y(a) = \alpha,\quad y(b) = \beta.$$

One supposes that the problem (D) has a unique solution.

THEOREM 6. *If the exact solution y of the problem (D) belongs to $W_2^4(\Delta_n)$ and $s_y \in \mathscr{S}_5(\Delta_n)$ is the function given by* (8), *then*

(19) $$\|y^{(k)} - s_y^{(k)}\|_\infty \le \sqrt{2}\cdot(b-a)^{2-k}\cdot\|\Delta_n\|^{3/2}\cdot\|y^{(4)}\|_2,\quad k = 0, 1, 2,$$

where

$$\|\Delta_n\| = \max\{t_{i+1} - t_i : i = \overline{0, n-1}\}.$$

Proof. Since $y''(t_i) - s_y''(t_i) = 0$, $i = \overline{0,n}$, by Rolle's theorem there exist $t_i^{(1)} \in (t_i, t_{i+1})$, $i = \overline{0, n-1}$ such that

$$y'''(t_i^{(1)}) - s_y'''(t_i^{(1)}) = 0,\quad i = \overline{0, n-1}.$$

Applying again Rolle's theorem, it follows the existence of the points $t_i^{(2)} \in (t_i^{(1)}, t_{i+1}^{(1)})$, $i = \overline{0, n-2}$ such that

$$y^{(4)}(t_i^{(2)}) - s_y^{(4)}(t_i^{(2)}) = 0,\quad i = \overline{0, n-2};$$

obviously that

$$|t_{i+1}^{(1)} - t_i^{(1)}| \le 2\|\Delta_n\|$$

and

$$|t_{i+1}^{(2)} - t_i^{(2)}| \le 3\|\Delta_n\|.$$

Since for every $t \in [a,b]$ there exists $i_0 \in \{0, 1, \dots, n-1\}$ such that $|t - t_{i_0}^{(1)}| \le 2\|\Delta_n\|$, one obtains:

$$|y'''(t) - s_y'''(t)| = \left|\int_{t_{i_0}^{(1)}}^{t} (y^{(4)}(u) - s_y^{(4)}(u))\,\mathrm{d}u\right| \le$$

$$\le \left|\int_{t_{i_0}^{(1)}}^{t} \mathrm{d}u\right|^{1/2} \cdot \left|\int_{t_{i_0}^{(1)}}^{t} [y^{(4)}(u) - s_y^{(4)}(u)]^2\,\mathrm{d}u\right|^{1/2} \le$$

$$\le \sqrt{2\|\Delta_n\|}\cdot\left|\int_{t_{i_0}^{(1)}}^{t} [y^{(4)}(u) - s_y^{(4)}(u)]\,\mathrm{d}u\right|^{1/2} =$$

$$= \sqrt{2\|\Delta_n\|}\cdot\|y^{(4)} - s_y^{(4)}\|_2 \le \sqrt{2}\cdot\|\Delta_n\|^{1/2}\cdot\|y^{(4)}\|_2$$

(the last inequality follows from Corollary 5, (18)).

It follows that

$$\|y''' - s_y'''\|_\infty \le \sqrt{2}\cdot\|\Delta_n\|^{1/2}\cdot\|y^{(4)}\|_2.$$

Similarly, for every $t \in [a,b]$, there exists $j_0 \in \{0, 1, \dots, n-2\}$ such that $|t - t_{j_0}| \le \|\Delta_n\|$, implying

$$\|y''(t) - s_y''(t)| = \left|\int_{t_{j_0}}^{t} [y'''(u) - s_y'''(u)]\,\mathrm{d}u\right| \le$$

Therefore

$$\|y'' - s_y''\|_\infty \le \sqrt{2}\,\|\Delta_n\|^{3/2}\cdot\|y^{(4)}\|_2,$$

showing that (19) holds for $k = 2$.

Taking into account the equalities $y(a) - s_y(a) = 0$ and $y(b) - s_y(b) = 0$, it follows (by Rolle's theorem) the existence of a point $c \in (a,b)$ such that $y'(c) - s_y'(c) = 0$. Then, for every $t \in [a,b]$ one has

$$|y'(t) - s_y'(t)| = \left|\int_{c}^{t} [y''(u) - s_y''(u)]\,\mathrm{d}u\right| \le$$

$$\le (b-a)\|y'' - s_y''\|_\infty \le \sqrt{2}\,(b-a)\|\Delta_n\|^{3/2}\,\|y^{(4)}\|_2,$$

showing that

$$\|y' - s_y'\|_\infty \le \sqrt{2}\cdot(b-a)\cdot\|\Delta_n\|^{3/2}\cdot\|y^{(4)}\|_2,$$

i.e., (19) holds for $k = 1$, too.

Finally, for every $t \in [a,b]$ one can write

$$|y(t) - s_y(t)| = \left|\int_{a}^{t} [y'(u) - s_y'(u)]\,\mathrm{d}u + y(a) - s_y(a)\right| =$$

$$= \left| \int_a^t [y'(u) - s_y'(u)]\,du \right| \leq (b-a)\,\|y' - s_y'\|_\infty \leq$$

$$\leq \sqrt{2}\cdot(b-a)\cdot\|\Delta_n\|^{3/2}\cdot\|y^{(4)}\|_2,$$

implying

$$\|y - s_y\|_\infty \leq \sqrt{2}\,(b-a)\cdot\|\Delta_n\|^{3/2}\cdot\|y^{(4)}\|_2. \quad \square$$

COROLLARY 7. *Under the hypotheses of Theorem 6 we have*

$$\lim_{\|\Delta_n\|\to 0} \|y^{(k)} - s_y^{(k)}\|_\infty = 0, \quad k = 0, 1, 2.$$

In the following, we shall approximate the exact solution y of the problem (D) by the function s_y given by

$$(20) \qquad s_y(t) = s_0(t)\cdot y(a) + s_1(t)\cdot y(b) + \sum_{k=0}^{n} S_k(t)\cdot y''(t_k), \ t\in[a,b].$$

This choice is motivated by the fact that the parameter $\varepsilon > 0$ is multiplied by y'' and s_y is determined by the interpolation conditions on the second derivative of y on the knots of Δ_n, which give $\varepsilon y''(t_i) = \varepsilon s_y''(t_i), i = \overline{0,n}$.

We shall use the following notations:

$$y(t_i) = y_i, \ y'(t_i) = y_i', \ i = \overline{0,n}$$

$$s_y(t_i) = u_i, \ s_y'(t_i) = u_i', \ i = \overline{0,n}$$

$$e(t) = y(t) - sy(t), \ e'(t) = y'(t) - s_y'(t), \ t\in[a,b]$$

$$e_i = y_i - u_i, \ e_i' = y_i' - u_i', \ i = \overline{0,n}.$$

Using the representation (20), one obtains

$$(21) \qquad u_i = s_0(t_i)\,\alpha + s_1(t_i)\,\beta + \sum_{k=0}^{n} S_k(t_i)\cdot f(t_k, y_k, y_k'), \ i = \overline{0,n}$$

$$u_i' = s_0'(t_i)\,\alpha + s_1'(t_i)\,\beta + \sum_{k=0}^{n} S_k'(t_i)\cdot f(t_k, y_k, y_k'), \ i = \overline{0,n}.$$

PROPOSITION 8. *If the real-valued function $f(t,u,v)$ defined on $D\subset[a,b]\times\mathbb{R}^2$ has continuous partial derivatives $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$, then the unknowns $u_i, u_i', i=\overline{0,n}$, can be obtained from the system*

$$(22) \qquad u_i = s_0(t_i)\,\alpha + s_1(t_i)\,\beta + \sum_{k=0}^{n} S_k(t_i)\cdot f(t_k, y_k, y_k'), \ i = \overline{0,n}$$

$$u_i' = s_0'(t_i)\,\alpha + s_1'(t_i)\,\beta + \sum_{k=0}^{n} S_k'(t_i)\cdot f(t_k, y_k, y_k'), \ i = \overline{0,n}.$$

Proof. By the hypotheses of the proposition we have

$$f(t_k, y_k, y_k') = f(t_k u_k + e_k, u_k' + e_k') = f(t_k, u_k, u_k') +$$

$$+ \frac{\partial f(t_k,\xi_k,\xi_k')}{\partial u} e_k + \frac{\partial f(t_k,\xi_k,\xi_k')}{\partial u} e_k', \ k = \overline{0,n},$$

where

$$\min(u_k, u_k + e_k) < \xi_k < \max(u_k, u_k + e_k), \ k = \overline{0,n}$$

$$\min(u_k', u_k' + e_k) < \xi_k' < \max(u_k', u_k' + e_k'), \ k = \overline{0,n}.$$

Replacing these in (21), we obtain the system:

$$u_i = s_0(t_i)\,\alpha + s_1(t_i)\,\beta + \sum_{k=0}^{n} S_k(t_i)\cdot f(t_k, u_k, u_k') + E_i, \ i = \overline{0,n}$$

$$u_i' = s_0'(t_i)\,\alpha + s_1'(t_i)\,\beta + \sum_{k=0}^{n} S_k'(t_i)\cdot f(t_k, u_k, u_k') + E_i', \ i = \overline{0,n},$$

having $2n+2$ equations and $2n+2$ unknowns $u_i, u_i', i = \overline{0,n}$.

By Theorem 6, the quantities

$$E_i = \sum_{k=0}^{n} S_k(t_i)\cdot\frac{\partial f(t_k,\xi_k,\xi_k')}{\partial u} e_k + \sum_{k=0}^{n} S_k(t_i)\cdot\frac{\partial f(t_k,\xi_k,\xi_k')}{\partial v} e_k',$$

$$E_i' = \sum_{k=0}^{n} S_k'(t_i)\cdot\frac{\partial f(t_k,\xi_k,\xi_k')}{\partial u} e_k + \sum_{k=0}^{n} S_k'(t_i)\cdot\frac{\partial f(t_k,\xi_k,\xi_k')}{\partial v} e_k'$$

have the order $O(\|\Delta_n\|^{3/2})$.

Eliminating $E_i, E_i', i = \overline{0,n}$, one obtains the system (22). $\square$

A NUMERICAL EXAMPLE

We consider the following singularly perturbed problem:

$$\begin{cases} -\varepsilon y''(x) + y'(x) = 0, & x \in [-1,1] \\ y(-1) = 1, \ y(1) = 0, \end{cases}$$

which may be regarded as a linearized one-dimensional version of a convection-dominated flow problem.

This problem has a unique solution $y(x) = \dfrac{e^{\frac{x+1}{\varepsilon}} - e^{\frac{2}{\varepsilon}}}{1 - e^{\frac{2}{\varepsilon}}}$, which displays one boundary layer at the point $x = 1$, of the length $O(\varepsilon)$.

Considering the solution $y_r(x)$ of the reduced problem

$$\begin{cases} y_r'(x) - 0, & x \in [-1,1] \\ y_r(-1) = 1, \end{cases}$$

the following estimations holds (see [6]):

$$|y(x) - y_r(x)| \le C(\varepsilon + e^{\frac{x-1}{\varepsilon}}), \ x \in [-1,1],$$

where C denotes an arbitrary constant independent of x and ε.

The exact solution is of the form

$$y(x) = y_r(x) + u(x),$$

where $u(x)$ will be approximated by

$$u(x) \approx \begin{cases} 0, & x \in [-1, 1 - p\varepsilon] \\ v(x), & x \in [1 - p\varepsilon, 1], \end{cases}$$

and the function $v(x)$ is the solution of the following problem:

$$\begin{cases} -\varepsilon v''(x) + v'(x) = 0, & x \in [1 - p\varepsilon, 1] \\ v(1 - p\varepsilon) = 0, & v(1) = -1. \end{cases}$$

Thus, we approximate the solution $y(x)$ by the solution $y_r(x)$ on the domain $[-1, 1 - p\varepsilon]$, so that the error would be $O(\varepsilon)$. In this way we obtain $p = \ln\frac{1}{\varepsilon}$. We approximate the solution $y(x)$ upon the domain $[1 - p\varepsilon, 1]$ by $y_r(x) + v(x)$.

We use elements from the space of spline functions $\mathscr{S}_{5(\Delta_n)}$ in order to approximate $v(x)$.

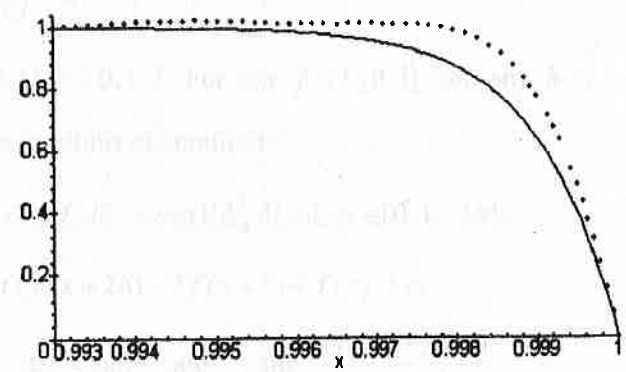
In Table 1 we present the error of approximation of solution $y(x)$ by $y_r(x) + s_v(x)$ upon the domain $[1 - p\varepsilon, 1]$ for different values of ε and n. The linear system for determining the spline s_v is solved by using a direct method.

Table 1

n/ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
3	$13 \cdot 10^{-2}$	$48 \cdot 10^{-3}$	$14 \cdot 10^{-2}$	–
8	–	$67 \cdot 10^{-4}$	$46 \cdot 10^{-4}$	–
12	–	–	$12 \cdot 10^{-4}$	$17 \cdot 10^{-3}$
20	–	–	–	$14 \cdot 10^{-3}$
50	–	–	–	$12 \cdot 10^{-3}$

In Fig. 1 there are displayed the exact solution $y(x)$ for $\varepsilon = 10^{-3}$ denoted by a continuous curve, and the approximated solution for $n = 3$ denoted by a dotted curve.

Fig. 1



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