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ON THE TRANSFORMATION OF THE SECOND ORDER MODULUS BY BERNSTEIN OPERATORS

RADU PĂLTĂNEA

Denote by $B_n : C[0,1] \to C[0,1]$ the well-known Bernstein operators given by

$$B_n(f)(x) := \sum_{l=0}^{n} p_{nl}(x) \cdot f\left(\frac{l}{n}\right),$$

where $p_{nl}(x) := \binom{n}{l} x^l (1-x)^{n-l}$, $f \in C[0,1], n \in \mathcal{N}, x \in [0,1]$. Let $e_i \in C[0,1]$, $e_i(x) := x^i$, $(x \in [0,1]), i = 0, 1, 2$. For any $f \in C[0,1]$ and any $h \in \left(0, \frac{1}{2}\right]$ consider the second order modulus of continuity:

$$\omega_2(f,h) := \sup\{|\Delta_h^2 f(x)|, x \in [0, 1-2h]\},$$

where $\Delta_h^2 f(x) := f(f(x+2h) - 2f(x+h) + f(x)$. Let

$$C := \sup_{n \in \mathcal{N}} \sup_{\substack{f \in C[0,1] \\ f \neq \text{linear}}} \sup_{h \in \left(0, \frac{1}{2}\right]} \frac{\omega_2(B_n(f), h)}{\omega_2(f,h)}.$$

C. Cottin and H. Gonska obtained in [2] that C is finite and, moreover, $C \leq 4.5$. This upper bound has been recently improved by J. Adell and A. Pérez-Palomares [1], in the form $C \leq 4$. On the other hand, D.-X. Zhou [5] obtained that $C > 1$. In fact in [5] it was proved the stronger result that Bernstein operators do not preserve the Lipschitz classes $Lip_2(\alpha, M)$, $\alpha \in (0,1]$. The aim of our paper is to give new improved estimates for the constant C.

THEOREM. *We have* $2 \leq C \leq 3$.

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Proof. Fix $n \in \mathcal{N}$, $h \in \left(0, \frac{1}{2}\right]$ and $x \in (0, 1-2h)$. We have for any $f \in C[0,1]$

(1) $$\Delta_h^2 B_n(f)(x) = \sum_{l=0}^{n} c_l \cdot f\left(\frac{l}{n}\right),$$

where

$$c_l := \Delta_h^2 p_{nl}(x), \ (0 \le l \le n).$$

Since $B_n(e_i) = e_i$, $(i = 0, 1)$, it follows

(2) $$\sum_{l=0}^{n} c_l = 0 \quad \text{and} \quad \sum_{l=0}^{n} l \cdot c_l = 0.$$

For $p, q \in (0,1)$ consider the function $\Psi(t) := (1+p)^t (1-q)^{n-t} + (1-p)^t (1+q)^{n-t} - 2$, $t \in [0, n]$. The following properties are immediate: $\Psi(0) > 0$, $\Psi(n) > 0$ and the derivative of Ψ is increasing on $[0, n]$. If we take $p := \frac{h}{x+h}$ and $q := \frac{h}{1-x-h}$, we have

$$c_l = \binom{n}{l} (x+h)^l (1-x-h)^{n-l} \Psi(l).$$

Then $c_0 > 0$ and $c_n > 0$. From (2) it follows that there is l, $0 < l < n$ such that $c_l < 0$. Hence there exist $t \in (0, n)$ such that $\Psi'(t) \le 0$, $(t \in [0, t_0])$ and $\Psi'(t) \ge 0$, $(t \in [0, t_0])$. From these it follows that there is the decomposition $\{0, 1, \ldots, n\} = I \cup J \cup K$ such that $i < j < k$ for all $i \in I$, $j \in J$, $k \in K$ and

$$c_i \ge 0, \ (i \in I), \quad c_j < 0, \ (j \in J), \quad c_k \ge 0, \ (k \in K).$$

Now denote

$$\Delta := \sum_{i \in I} c_i \cdot \sum_{k \in K} k c_k - \sum_{i \in I} i c_i \cdot \sum_{k \in K} c_k.$$

We have that $\sum_{i \in I} i c_i \big/ \sum_{i \in I} c_i$ belongs to the convex hull of the set I. Then it is smaller than $\sum_{k \in K} k c_k \big/ \sum_{k \in K} c_k$ which belongs to the convex hull of the set K. Consequently, we have $\Delta > 0$.

For any $j \in J$ denote

$$u_j := \frac{1}{\Delta} \sum_{k \in K} (k-j) c_k, \quad \text{and} \quad v_j := \frac{1}{\Delta} \sum_{i \in I} (j-i) c_i$$

and consider the linear positive functional $G_j : C[0,1] \to \mathcal{R}$, given by

$$G_j(f) := u_j \sum_{i \in I} c_i \cdot f\left(\frac{i}{n}\right) + v_j \sum_{k \in K} c_k \cdot f\left(\frac{k}{n}\right), \quad (f \in C[0,1]).$$

One immediately obtains that $G_j(e_0) = 1$ and $G_j(e_1) = \frac{j}{n}$. In [4] we obtained the following estimate: If $G : C[I] \to \mathcal{R}$ is a linear positive functional, if I is a closed interval, if $x \in I$ and if $G(e_0) = 1$ and $G(e_1) = x$, then we have $|G(f) - f(x)| \le \le \left[1 + \frac{1}{2} h^{-2} G((e_1 - x e_0)^2)\right] \omega_2(f, h)$, for any $f \in C(I)$ and any $h > 0$. Consequently, we have

(3) $$\left| G_j(f) - f\left(\frac{j}{n}\right) \right| \le$$

$$\le \left[1 + \frac{1}{2} h^{-2} G_j\left(\left(e_1 - \frac{j}{n} e_0\right)^2\right)\right] \omega_2(f, h), \quad (f \in C[0,1]).$$

On the other hand, from (1) and (2) one can derive the following representation:

(4) $$\Delta_h^2 B_n(f)(x) = \sum_{j \in J} (-c_j) \left[G_j(f) - f\left(\frac{j}{n}\right)\right], \ (f \in C[0,1]).$$

Also, we have

$$\sum_{j \in J} (-c_j) \le \sum_{j \in J} 2 \cdot p_{nj}(x+h) \le 2.$$

Therefore we obtain

$$|\Delta_h^2 B_n(f)(x)| \le \left[2 + \frac{1}{2} h^{-2} \sum_{j \in J} (-c_j)\, G_j\left(\left(e_1 - \frac{j}{n} e^0\right)^2\right)\right] \omega_2(f, h),$$

$$(f \in C[0,1]).$$

We have $G_j(e_2) - e_2\left(\frac{j}{n}\right) = G_j\left(\left(e_1 - \frac{j}{n} e_0\right)^2\right)$. Then, by using the well-known relation $B_n(e_2)(x) = x^2 + \frac{x(1-x)}{n}$, we deduce from (4)

$$\sum_{j \in J} (-c_j) G_j\left(\left(e_1 - \frac{j}{n} e_0\right)^2\right) = \Delta_h^2 B_n(e_2)(x) = 2h^2 \frac{n-1}{n} \le 2h^2.$$

Consequently, we have

$$|\Delta_h^2 B_n(f)(x)| \le 3\cdot\omega_2(f,h),$$

$$\left(f\in C[0,1],\ h\in\left(0,\frac{1}{2}\right),\ x\in(0,1-2h)\right).$$

This last inequality can be extended by passing to limit, for all $h\in\left[0,\frac{1}{2}\right]$ and $x\in[0,1-2h]$. Therefore we have proved the inequality

$$\omega_2 B_n(f)\, h) \le 3\cdot\omega_2(f,h),\quad f\in C[0,1],\ h\in[0,1].$$

Conversely, for any integer $n\ge 1$ consider the function $f_n\in C[0,1]$, defined as follows: $f_n(0)=0$, $f_n(1)=0$, $f_n(k\cdot 2^{-j})=1-2^{-j}$, if $1\le j\le n$ and k is odd such that $1\le k\le 2^j-1$ and f_n is linear on each of the intervals of the form $[(i-1)\,2^{-n}, i2^{-n}]$.

We have

$$\omega_2\left(f_n,\frac{1}{2}\right)=1,\ (n\ge 1). \tag{5}$$

Indeed, let the function $g(x,h):=|\Delta_h^2 f_n(x)|$, $h\in\left[0,\frac{1}{2}\right]$, $x\in[0,1-2h]$. We have $g\left(0,\frac{1}{2}\right)=1$. Let (x_0,h_0) be a point in which g reaches its maximum. Since g is a piecewise linear function with regard to each of his arguments, it follows that we can consider that at least two of the points x_0, x_0+h_0, x_0+2h_0 belong to the set $M:=\{k\cdot 2^{-n} | 0\le k\le 2^n\}$. Since $f_n(t)\in\left[\frac{1}{2},1\right]$ when $t\in[2^{-n},1-2^{-n}]$, we have $g(x_0,h_0)\le 1$ when $x_0, x_0+2h_0\in[2^{-n},1-2^{-n}]$. Then it remains to consider the case where $x_0=0$ or $x_0+2h_0=1$. Let, for example, $x_0=0$. If we suppose $x_0+h_0\notin M$, we have $x_0+2h_0=k\cdot 2^{-n}$ with k odd. Consequently, $g(x_0,h_0)=$ $=|0-2f_n(x_0+h_0)+1-2^{-n}|<1$, which is impossible. Then it follows $x_0=0$, $x_0+h_0=k\cdot 2^{-j}$ and $x_0+2h_0=k\cdot 2^{1-j}$, where $1\le j\le n$ and k is odd. In this case $g(x_0,h_0)=1$. Then (5) is true.

Let $n\ge 1$ and denote $q:=2^{2n}$. We have $B_q(f_{2n})(0)=B_q(f_{2n})(1)=0$ and

$$B_q(f_{2n})\left(\frac{1}{2}\right)=2^{-q}\sum_{k=0}^{q}\binom{q}{k}f_{2n}\left(\frac{k}{q}\right)>2^{-q}\sum_{\substack{0\le k\le q\\ 2^2\nmid k}}\binom{q}{k}f_{2n}\left(\frac{k}{q}\right)\ge$$

$$\ge 2^{-q}\sum_{\substack{0\le k\le q\\ 2^2\nmid k}}\binom{q}{k}(1-2^{-n-1})=2^{-q}(1-2^{-n-1})\left[2^q-\sum_{j=0}^{2^n}\binom{q}{j\cdot 2^n}\right].$$

We have $\binom{q}{j\cdot 2^n}\le 2^{-n}\sum_{k=j\cdot 2^n}^{(j+1)2^n-1}\binom{q}{k}$ for $0\le j<2^{n-1}$ and hence $\sum_{j=0}^{2^n}\binom{q}{j\cdot 2^n}=$

$$=2\sum_{j=0}^{q/2-1}\binom{q}{j\cdot 2^n}+\binom{q}{q/2}\le 2^{1-n}\sum_{k=0}^{q/2-1}\binom{q}{k}+\binom{q}{q/2}\le 2^{q-n}+\binom{q}{q/2}.$$

Therefore

$$|\Delta_{1/2}^2 B_q(f_{2n})(0)|\ge 2(1-2^{-n-1})\left(1-2^{-n}-\binom{q}{q/2}\cdot 2^{-q}\right).$$

Since $\lim_{q\to\infty}\binom{q}{q/2}\cdot 2^{-q}=0$, it follows

$$\sup_{n\in\mathbb{N}}|\Delta_{1/2}^2 B_{2^{2n}}(f_{2n})(0)|=2$$

and, therefore, the left-side inequality in the theorem is also proved.

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Radu Păltănea, Deptartment of Mathematics
Transilvania University,
29, Eroilor Str. RO. 2200 Braşov
Romania