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ON THE DYNAMIC PROGRAMMING PRINCIPLE FOR OPTIMAL CONTROL PROBLEMS

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1. INTRODUCTION

We consider the optimal control problem of Bolza, formulated by M. R. Hestenes in [1]:

Minimize

$$I^0(x,u,b)=g^0(b)+\int_{t_0}^{t_1} L^0(t,x(t),u(t),b)\,dt, \tag{1}$$

when the functions $x=(x^1,x^2,\dots,x^n), u=(u^1,u^2,\dots,u^q)$ and the parameter $b=(b^1,b^2,\dots,b^r)$ are subject to

$$x^{\prime i}(t)=f^i(t,x(t),u(t),b),\quad t\in[t^0,t^1],\ 1\le i\le n,$$

$$h^\alpha(t,x(t),u(t),b)\le 0,\quad t\in[t^0,t^1],\ 1\le\alpha\le m',$$

$$h^\alpha(t,x(t),u(t),b)=0,\quad t\in[t^0,t^1],\ m'<\alpha\le m,$$

$$t^s=T^s(b),\ x^i(t^s)=X^{is}(b),\quad 1\le i\le n,\quad s=0,1,$$

$$I^\gamma(x,u,b)\le 0,\quad 1\le\gamma\le p',$$

$$I^\gamma(x,u,b)=0,\quad p'<\gamma\le p,$$

where

$$I^\gamma(x,u,b)=g^\gamma(b)+\int_{t^0}^{t_1} L^\gamma(t,x(t),u(t),b)\,dt,\quad 1\le\gamma\le p.$$

This problem will be denoted by P.

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A system $(u(t), x(t), b)$, $t^0 \le t \le t^1$, simply denoted by (u, x, b), which satisfies the conditions (2) will be called *admissible strategy* for the Problem P. The class of all admissible strategies will be denoted by A. An admissible strategy $(u_*(t), x_*(t), b_*), t_*^1 \le t \le t_*^1$ which minimizes I^0 is called *optimal strategy*.

M. R. Hestenes [1] presents several properties of the optimal strategy (first-order necessary conditions and a maximum principle) in the following hypotheses:

1° The state functions x^i are continuous, the control functions u^k are piecewise continuous on $[t_*^0, t_*^1]$.

2° All functions used $f^i, h^\alpha, T^s, X^{is}, g^\gamma, L^\gamma$ are of class C^1 on $[t_*^0, t_*^1]$.

3° Rank $\left(\dfrac{\partial h^\alpha}{\partial u^k}, \delta_{\alpha\beta} h^\beta\right)_{m\times(m+q)} = m$, on $[t_*^0, t_*^1]$,

where $\alpha = 1, 2, \ldots, m$ denotes the row index and $k = 1, 2, \ldots, q$, $\beta = 1, 2, \ldots, m$ are column indices.

The results presented in the next paragraph permit the use of those properties such that the conditions 3° or 2° are not fulfilled on the whole interval $[t_*^0, t_*^1]$.

2. THE DYNAMIC PROGRAMMING PRINCIPLE FOR OPTIMAL CONROL PROBLEMS

Firstly, we consider the following problem:

Minimize

$$I^0(x, u) = \int_{t^0}^{t^1} L^0(t, x(t), u(t))\,dt$$

subject to

$$x^{i\prime}(t) = f^i(t, x(t), u(t))\,dt, \quad t \in [t^0, t^1],\ 1 \le i \le n,$$

$$h^\alpha(t, x(t), u(t)) \le 0, \quad t \in [t^0, t^1],\ 1 \le \alpha \le m',$$

$$h^\alpha(t, x(t), u(t)) = 0, \quad t \in [t^0, t^1],\ m' < \alpha \le m,$$

$$t^s = T^s(b), x^i(t^s) = X^{is}(b), \quad 1 \le i \le n, \quad s = 0, 1.$$

Denote this problem by P_1 and observe that it represents a particular case of Problem P.

LEMMA 2.1. *Let* $(u_*(t), x_*(t), b)$, $t^0 \le t \le t^1$, *be an optimal strategy for Problem* P_1. *If* $[t_1, t_2] \subset [t_*^0, t_*^1]$ *and* $(u(t), x(t)), t_1 \le t \le t_2$ *is admissible strategy for Problem* P_1 *on the interval* $[t_1, t_2]$ *with* $x(t_1) = x_*(t_1)$ *and* $x(t_2) = x_*(t_2)$, *then* $(u_0(t), x_0(t), b_*)$, $t_*^0 \le t \le t_*^1$ *defined by*

$$u_0(t) = \begin{cases} u_*(t), t_*^0 \le t \le t_1, \\ u(t), t_1 < t < t_2, \\ u_*(t), t_2 \le t \le t_*^1, \end{cases} \qquad x_0(t) = \begin{cases} x_*(t), t_*^0 \le t \le t_1, \\ x(t), t_1 < t < t_2, \\ x_*(t), t_2 \le t \le t_*^1, \end{cases} \tag{7}$$

is an admissible strategy.

Proof. Evidently (u_0, x_0, b_*) satisfies the restraints of Problem P_1. □

The next theorems express the optimality property on any interval $[t_1, t_2] \subset [t_*^0, t_*^1]$ of an optimal strategy $(u_*(t), x_*(t), b_*)$, $t_*^0 \le t \le t_*^1$.

THEOREM. 2.2. *If* $(u_*(t), x_*(t), b_*)$, $t_*^0 \le t \le t_*^1$ *is an optimal strategy for Problem* P_1, *then this is an optimal strategy on any interval with fixed extremities* $[t_1, t_2] \subset [t_*^0, t_*^1]$.

Proof. Suppose that there exists an interval $[t_1, t_2] \subset [t_*^0, t_*^1]$ on which the restriction of (u_*, x_*, b_*) is not optimal. Then there is an admissible strategy $(u(t), x(t))$, $t_1 \le t \le t_2$ with $x(t_1) = x_*(t_1), x(t_2) = x_*(t_2)$ and

$$\int_{t_1}^{t_2} L^0(t, x(t), u(t))\,dt < \int_{t_1}^{t_2} L^0(t, x_*(t), u_*(t))\,dt.$$

Considering the strategy $(u_0(t), x_0(t), b_*)$, $t_*^0 \le t \le t_*^1$ given by (7), this is an admissible strategy. Also, we have

$$\int_{t_*^0}^{t_*^1} L^0(t, x_0(t), u_0(t))\,dt < \int_{t_*^0}^{t_*^1} L^0(t, x_*(t), u_*(t))\,dt,$$

which contradicts the optimality of (u_*, x_*, b_*). □

THEOREM. 2.3. *Let* $(u_*(t), x_*(t), b_*)$, $t_*^0 \le t \le t_*^1$ *be an optimal strategy for Problem* P. *Then the restriction of* (u_*, x_*, b_*) *on any interval with fixed extremities* $[t_1, t_2] \subset [t_*^0, t_*^1]$, *is an optimal strategy for the following problem, denoted by* P'.

Minimize

$$\frac{t_2 - t_1}{T^1(b) - T^0(b)} g^0(b) + \int_{t_1}^{t_2} L^0(t, x(t), u(t), b)\,dt$$

subject to

$$\dot{x}^i(t) = f^i(t, x(t), u(t), b), \quad t \in [t_1, t_2], \quad 1 \le i \le n,$$

$$h^\alpha(t, x(t), u(t), b) \le 0, \quad t \in [t_1, t_2], \quad 1 \le \alpha \le m',$$

$$h^\alpha(t, x(t), u(t), b) = 0, \quad t \in [t_1, t_2], \quad m' < \alpha \le m,$$

$$x^i(t_1) = X^{i0}(b) + K^{i0}, \quad 1 \le i \le n,$$

$$x^i(t_2) = X^{i1}(b) + K^{i1}, \quad 1 \le i \le n,$$

$$y^\gamma(x, u, b) \le 0, \quad 1 \le \gamma \le p',$$

$$y^\gamma(x, u, b) = 0, \quad p' < \gamma \le p,$$

where

$$y^\gamma(x, u, b) = g^\gamma(b) + K^\gamma + \int_{t_1}^{t_2} L^\gamma(t, x(t), u(t), b)\,\mathrm{d}t, \ 1 \le \gamma \le p$$

and the constants K^{is}, K^γ *are given by*

$$K^{is} = \int_{t_*^s}^{t_s} f^i(t, x_*(t), u_*(t), b_*)\,\mathrm{d}t, \ 1 \le i \le n, \ s = 0, 1,$$

$$K^\gamma = \int_{t_*^0}^{t_1} L^\gamma(t, x_*(t), u_*(t), b_*)\,\mathrm{d}t + \int_{t_2}^{t_*^1} L^\gamma(t, x_*(t), u_*(t), b_*)\,\mathrm{d}t, \ 1 \le \gamma \le p.$$

Proof. We introduce the functions

$$y = (y^1, y^2, \dots, y^r), \quad z = (z^1, z^2, \dots, z^p), \quad w = (w^1, w^2, \dots, w^{p'})$$

by

$$y^j(t) = b^j, \quad 1 \le j \le r,$$

$$z^\gamma(t) = \int_{t^0}^{t} L^\gamma(t, x(t), u(t), y(t))\,\mathrm{d}t, \quad 1 \le \gamma \le p,$$

$$w^\gamma(t) = z^\gamma(t^1), \quad 1 \le \gamma \le p',$$

for any $t \in [t^0, t^1]$.

Then Problem P can be written as follows:

Minimize

$$\int_{t^0}^{t^1} \left(L^0(t, x(t), u(t)) + \frac{g^0(y(t))}{T^1(y(t)) - T^0(y(t))} \right) \mathrm{d}t$$

subject to

$$\dot{x}^i(t) = f^i(t, x(t), y(t), u(t)), \quad 1 \le i \le n,$$

$$\dot{y}^j(t) = 0, \quad 1 \le j \le r,$$

$$\dot{z}^\gamma(t) = L^\gamma(t, x(t), y(t), u(t)), \quad 1 \le \gamma \le p,$$

$$\dot{w}(t) = 0, \quad 1 \le \gamma \le p',$$

$$h^\alpha(t, x(t), y(t), u(t)) \le 0, \quad 1 \le \alpha \le m',$$

$$h^\alpha(t, x(t), y(t), u(t)) \le 0, \quad m' \le \alpha \le m,$$

$$g^\gamma(y(t)) + w^\gamma(t) \le 0, \quad 1 \le \gamma \le p',$$

$$t^s = T^s(b), \quad s = 0, 1,$$

$$x^i(t^s) = X^{is}(b), \quad 1 \le i \le n, \quad s = 0, 1,$$

$$y^j(t^s) = b^j, \quad 1 \le j \le r, \quad s = 0, 1,$$

$$z^\gamma(t^0) = 0, \quad 1 \le \gamma \le p,$$

$$z^\gamma(t^1) = c^\gamma, \quad 1 \le \gamma \le p',$$

$$z^\gamma(t^1) = -g^\gamma(b), \quad p' < \gamma \le p,$$

$$w^\gamma(t^s) = c^\gamma, \quad 1 \le \gamma \le p', \quad s = 0, 1,$$

for all $t \in [t^0, t^1]$.

This is a P_1 problem and, in accordance with Theorem 2.2, the restriction of (u_*, x_*, b_*) on any interval $[t_1, t_2] \subset [t_*^0, t_*^1]$ is an optimal strategy. Writing the last problem on the interval $[t_1, t_2]$ and returning to the former notations, we obtain Problem P'. □

Remark 2.4. Problems P and P' have the same Hamiltonian [1]; consequently, all necessary conditions, excepting the transversality condition, have the same form. The transversality condition for Problem P' differs, as $g^0(b)$ becomes $\dfrac{t_2 - t_1}{T^1(b) - T^0(b)} g^0(b)$.

3. COMMENTARIES

1. Taking $t_2 = t_*^1$, the similar particular result given by Şt. Mirică [2] is obtained.

2. As it will be shown paper, the above results are useful for solving the optimal control problem of mechanical systems with velocity restraints.

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