REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
TOME XXVII, N^{o} 2, 1998, pp. 339–350

A ONE-STEP SPLINE METHOD FOR THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

ADRIAN REVNIC

1. INTRODUCTION

The spline numerical methods are an accurate instrument for obtaining global approximations of the solutions of ODEs and their derivatives. Due to their high order, they often fail to have good stability properties, so they become useless for an important class of ODEs such as the stiff problems. In [5], it is constructed a spline method which has stability properties that make it suitable for solving stiff problems (stiff stability). In the following, an A-stable method of a similar type is obtained.

Consider the initial value problem

(1) $$y'(t) = f(t, y(t))$$
$$y(0) = y_0$$

for $t \in [0, T]$, $f \in C^p([0, T] \times \mathbb{R}^m, \mathbb{R}^m)$, $p \geq 0$. Assume that f satisfies the Lipschitz conditions

(2) $$\|f^{(q)}(t, u) - f^{(q)}(t, v)\| \leq L\|u - v\|$$

for all $t \in [0, T]$, $u, v \in \mathbb{R}^m$ and $q = 0, \ldots, p$.

In (2) we define $f^{(q)}$ in the following way:

$$f^{(q+1)} := f_x^{(q)} + f_x^{(q)} f, \quad q = 0, \ldots, p.$$
$$f^{(0)} := f.$$

Define on $[0, T]$ the uniform mesh

$$\Delta : 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T,$$

AMS subject classification: 65L10, 65Q05.

$n \geq 1$ with the step $h := t_{k+1} - t_k$ for all $k = 0, \ldots, n-1$. Denote $y_k^{(j)} := y^{(j)}(t_k)$ for $k = 0, \ldots, n$ and $j = 0, \ldots, p$.

2. THE FIRST APPROXIMATION PROCESS

Let y be the exact solution of the problem. By integrating from t_k to t we get

(3) $$y(t) = y_k + \int_{t_k}^{t} f(x, y(x))\,\mathrm{d}x$$

and for $t := t_{k+1}$

$$y_{+1}(t) = y_k + \int_{t_k}^{t_{k+1}} f(x, y(x))\,\mathrm{d}x.$$

We consider $\{\overline{y}_k^{(j)} \mid k = 0, \ldots, n;\ j = 0, \ldots, p+1\}$, approximations for $\{y^{(j)}(t_k) \mid k = 0, \ldots, n;\ j = 0, \ldots, p+1\}$ defined in the following way:

(4) $$\overline{y}_{k+1} = \overline{y}_k + \int_{t_k}^{t_{k+1}} f(x, \overline{H}_{k+1}(x))\,\mathrm{d}x,$$

where $\overline{H}_{k+1}$ is the full Hermite-interpolation polynomial for the knots t_k, t_{k+1} and the values $\{\overline{y}_k^{(j)}\}_{j=0,\ldots,p+1}$, respectively $\{\overline{y}_{k+1}^{(j)}\}_{j=0,\ldots,p+1}$. $\overline{H}_{k+1}$ is a polynomial of degree $2p+3$ and for $x \in [t_k, t_{k+1}]$ it can be written in the following form:

(5) $$\overline{H}_{k+1}(t) = \sum_{j=0}^{p+1} g_{kj}(t)\,\overline{y}_k^{(j)} + \sum_{j=0}^{p+1} g_{k+1j}(t)\,\overline{y}_{k+1}^{(j)},$$

where the full Hermite fundamental polynomials g_{ij} are defined by

(6) $$g_{ij}^{(l)}(t_m) = \delta_{i,m-k}\,\delta_{jl},\ i = 0, 1,\ m = k,\ k+1 \text{ and } j, l = 0, \ldots, r+1.$$

Let

$$w(t) = (t - t_k)^{p+2}(t - t_{k+1})^{p+2}$$

$$u_i(t) = \frac{w(t)}{(t - t_i)^{p+2}},\quad i = k, k+1.$$

The $2p+3$ degree interpolator becomes

$$\overline{H}_{k+1}(t) = \sum_{i=k}^{k+1}\sum_{j=0}^{p+1} g_{ij}(t)\,\overline{y}_i^{(j)} =: \sum_{i=k}^{k+1} g_i(t).$$

Now we can consider the following Taylor expansions for the polynomials $\frac{g_i(\cdot)}{u_i(\cdot)}$:

(8) $$\frac{g_i(t)}{u_i(t)} = \sum_{j=0}^{p+1}\frac{(t-t_i)^j}{j!}\,\frac{d^j}{dt^j}\left(\frac{g_i(t)}{u_i(t)}\right)\Bigg|_{t=t_i}.$$

Substituting (8) in (7) and taking into account (6), (7) becomes

(9) $$\overline{H}_{k+1}(t) = \sum_{i=k}^{k+1} u_i(t)\sum_{j=0}^{p+1}\frac{(t-t_i)^j}{j!}\sum_{l=0}^{j}\binom{j}{l}\overline{y}_i^{(l)}\left(\frac{1}{u_i(t)}\right)^{(j-l)}$$

and, after rearranging the summation order and indices in (9), we obtain

(10) $$\overline{H}_{k+1}(t) = \sum_{i=k}^{k+1}\sum_{j=0}^{p+1}\left(u_i(t)\sum_{l=j}^{p+1}\binom{j}{l}\frac{(t-t_i)^l}{l!}\,\frac{d^{l-j}}{dt^{l-j}}\left(\frac{1}{u_i(t)}\right)\Bigg|_{t=t_i}\right)\overline{y}_i^{(j)},$$

Comparing (7) and (10), we get an expression for the full Hermite fundamental polynomials

$$g_{ij}(t) = u_i(t)\sum_{l=j}^{p+1}\binom{l}{j}\frac{(t-t_i)^l}{l!}\,\frac{d^{l-j}}{dt^{l-j}}\left(\frac{1}{u_i(t)}\right)\Bigg|_{t=t_i}$$

or, in an equivalent formulation,

(11) $$g_{ij}(t) = \frac{w(t)}{j!}\sum_{l=1}^{p+2-j}\frac{(t-t_i)^{-l}}{(p+2-j-l)!}\,D_l^{p+2-j-l}\left(\frac{(t-t_i)^{p+2}}{w(t)}\right)\Bigg|_{x=t_i}$$

with $D_l := \frac{d}{dt}$, $i = k,\ k+1$, $j = 0, \ldots, p+1$.

We define the approximations $\overline{y}_{k+1}^{(j+1)}$, for $j = 0, \ldots, r$, in the following way:

(12) $$\overline{y}_{k+1}^{(j+1)} = f^{(j)}(t_{k+1}, \overline{y}_{k+1}),\quad j = 0, \ldots, r.$$

The equations (4) and (12) form the first approximation process.

Obviously, $\overline{y}_0^{(j)} = y_0^{(j)}$, for $j = 0, \ldots, r+1$.

THEOREM 1. *The implicit non-linear system in* $\overline{y}_{k+1}$ *given by* (4)–(12) *is uniquely solvable, provided h is sufficiently small.*

Proof. Using (4)–(12), the system becomes

$$\overline{y}_{k+1} = \overline{y}_k + $$
$$+\int_{t_k}^{t_{k+1}} f\left(x, \sum_{j=0}^{p+1} g_{kj}(x)\,\overline{y}_k^{(j)} + g_{k+1,0}(x)\,\overline{y}_{k+1} + \sum_{j=1}^{p+1} g_{k+1j}(x)\,f^{(j)}(t_{k+1}, \overline{y}_{+1})\right) dx.$$

Consider the application $\mathcal{A}: \mathbb{R}^m \to \mathbb{R}^m$,

$$\mathcal{A}(u) = \overline{y}_k + $$
$$+\int_{t_k}^{t_{k+1}} f\left(x, \sum_{j=0}^{p+1} g_{kj}(x)\,\overline{y}_k^{(j)} + g_{k+1,0}(x)\,u + \sum_{j=1}^{p+1} g_{k+1j}(x)\,f^{(j)}(t_{k+1}, u)\right) dx.$$

Then the system becomes $u = \mathcal{A}u$. Consider $u, v \in \mathbb{R}^m$. Applying the norm properties and the Lipschitz conditions (2), we obtain

$$|\mathcal{A}u - \mathcal{A}v|| \le L \int_{t_k}^{t_{k+1}} |g_{k+1,0}(x)|\,||u-v||\,dx +$$

$$+L\int_{t_k}^{t_{k+1}} \sum_{j=1}^{p+1} |g_{k+1j}(x)|\,||f^{(j)}(t_{k+1}, u) - f^{(j)}(t_{k+1}, v||\,dx \le$$

$$\le Lh\,||g_{k+1,0}||_\infty\,||u-v|| + L^2 h \sum_{j=1}^{p+1} ||g_{k+1j}||_\infty\,||u-v|| \le$$

$$\le \left(Lh\,||g_{k+1,0}||_\infty + L^2 h \sum_{j=1}^{p+1} ||g_{k+1j}||_\infty\right) ||u-v||.$$

By $||\cdot||_\infty$ we intend the uniform functional norm on the interval $[t_k, t_{k+1}]$. Denote

$$L_A = \left(||g_{k+1,0}||_\infty + L\sum_{j=1}^{p+1} ||g_{k+1j}||_\infty\right) Lh,$$

so we get

$$||\mathcal{A}u - \mathcal{A}v|| \le L_{\mathcal{A}}\,||u-v||.$$

We only have to prove now that $||g_{ij}||_\infty \le c_{ij}$, where c_{ij} are constants independent of h. Using (11), for $i = k,\ k+2,\ j = 0, \dots, p+1$

$$||g_{ij}||_\infty \le \frac{h^{2p+4}}{j!} \sum_{l=1}^{p+2-j} \frac{h^{-l}}{(p+2-j-l)!}(p+2)\dots(2p+3-j-l)\,h^{-2p-4+j+l} \le$$

$$\le \frac{h^j}{j!} \sum_{l=1}^{p+2-j} \frac{(2p+3-j-l)!}{(p+1)!(p+2-j-l)!} \le \frac{T^j}{j!} \sum_{l=1}^{p+1-j} \frac{(2p+2-j-l)!}{(p+1)!(p+1-j-l)!}.$$

Using the previous estimate and (13), we see that for h small enough $L_{\mathcal{A}} < 1$. So $\mathcal{A}$ is a contraction and system (4)–(12) is uniquely solvable, following the Picard-Banach fixed point theorem. □

LEMMA 2. *The inequality*

$$||y_{k+1} - \overline{y}_{k+1}|| \le \frac{1+c_1 h}{1-c_2 h}\,||y_k - \overline{y}_k|| + \frac{LM}{1-c_2 h} h^{r+2}$$

holds for $k = 0, \dots, n-1$, *where* M, c_1 *and* c_2 *are positive constants, independent of* h.

Proof. Denote by H_{k+1} the full Hermite-interpolation polynomial for the knots t_k, t_{k+1} and the values $\{y_k^{(j)}\}_{j=0,\dots,p+1}$, respectively $\{y_{k+1}^{(j)}\}_{j=0,\dots,p+1}$. For $y \in C^{p+1}([0, T])$, using a result from the theory of interpolation,

(14) $$||y(x) - H_{k+1}(x)|| \le MH^{p+1}, \quad x \in [t_k, t_{k+1}]$$

with M a positive constant, independent of h. So, using (2), (3) and (14), we get the estimates

$$||y_{k+1} - \overline{y}_{k+1}|| \le ||y_k - \overline{y}_k|| + \int_{t_k}^{t_{k+1}} ||f(x, y(x)) - f(x, \overline{H}_{k+1}(x))||\,dx \le$$

$$\le ||y_k - \overline{y}_k|| + L\int_{t_k}^{t_{k+1}} ||y(x)) - \overline{H}_{k+1}(x))||\,dx \le$$

$$\le ||y_k - \overline{y}_k|| + L\int_{t_k}^{t_{k+1}} ||y(x)) - H_{k+1}(x))||\,dx +$$

$$+\int_{t_k}^{t_{k+1}} ||H_{k+1}(x)) - H_{k+1}(x))||\,dx \le ||y_k - \overline{y}_k|| + LMh^{p+2} +$$

$$+\int_{t_k}^{t_{k+1}}\left\|\sum_{j=0}^{p+1} g_{kj}(x)\,(y_k^{(j)}-\bar{y}_k^{(j)})+\sum_{j=0}^{p+1} g_{k+1j}(x)\,(y_{k+1}^{(j)}-\bar{y}_{k+1}^{(j)})\right\|dx\leq$$

$$\leq\|y_k-\bar{y}_k\|+LMh^{p+2}+$$

$$+\|y_k-\bar{y}_k\|\,h\,\|g_{k,0}\|_\infty+\|y_k-\bar{y}_k\|\,Lh\sum_{j=1}^{p+1}\|g_{kj}\|_\infty+$$

$$+\|y_{k+1}-\bar{y}_{k+1}\|\,h\,\|g_{k+1,0}\|_\infty+\|y_{k+1}-\bar{y}_{k+1}\|\,Lh\sum_{j=1}^{p+1}\|g_{k+1j}\|_\infty\leq$$

$$\leq\|y_k-\bar{y}_k\|+c_1h\|y_k-\bar{y}_k\|+c_2h\|y_{k+1}-\bar{y}_{k+1}\|+LMh^{p+2},$$

or, for h small enough,

$$\|y_{k+1}-\bar{y}_{k+1}\|\leq\frac{1+c_1h}{1-c_2h}\|y_k-\bar{y}_k\|+\frac{LM}{1-c_2h}h^{p+2}.\ \square$$

Remark. The previous result shows that the method has the local order $p+2$.

THEOREM 3. *The inequality*

$$\|y_{k+1}^{(j)}-\bar{y}_{k+1}^{(j)}\|\leq c_3h^{p+1}$$

holds for all $k=0,\ldots,n-1,\ j=0,\ldots,r+1$, *where* c_3 *is a positive constant, independent of* h.

Proof. For $j=0$, from Lemma 2.1, multiplying by a suitable factor, we get for all $i\in 0,\ldots,k,\ k=1,\ldots,n-1$, that

$$\left(\frac{1+c_1h}{1-c_2h}\right)^{k-i}\|y_{i+1}-\bar{y}_{i+1}\|\leq\left(\frac{1+c_1h}{1-c_2h}\right)^{k-i+1}\|y_i-\bar{y}_i\|+\frac{LM(1+c_2h)^{k-i}}{(1-c_2h)^{k-i+1}}h^{p+2}$$

And, by summing the previous inequalities from $i=1$ to $i=k$, we get

$$\|y_{k+1}-\bar{y}_{k+1}\|\leq LM\sum_{i=1}^{k+1}\frac{(1+c_1h)^{i-1}}{(1-c_2h)^i}h^{p+2}=$$

$$\leq LM\frac{1}{1-c_2h}\,\frac{\left(\frac{1+c_1h}{1-c_2h}\right)^{k+1}}{\frac{1+c_1h}{1-c_2h}-1}h^{p+2}=LM(c_1+c_2)\left(\left(\frac{1+c_1h}{1-c_2h}\right)^{k+1}-1\right)h^{p+1}.$$

We estimate

$$\left(\frac{1+c_1h}{1-c_2h}\right)^{k+1}=\left(1+\frac{(c_1+c_2)\,h}{1-c_2h}\right)^{k+1}\leq$$

$$\leq\left[\left(1+\frac{(c_1+c_2)\,T}{1-c_2T}\right)^{\frac{n-c_2T}{(c_1+c_2)T}}\right]^{\frac{(c_1+c_2)T\,n}{n-c_2T}}\leq e^{(c_1+c_2)T}$$

and taking $c_3:=LM(c_1+c_2)\,(e^{(c_1+c_2)T}-1)$ we have

$$\|y_{k+1}-\bar{y}_{k+1}\|\leq c_3h^{p+1}.$$

Using (12) and the Lipschitz conditions (2), we obtain

$$\|y_{k+1}^{(j)}-\bar{y}_{k+1}^{(j)}\|\leq\|f^{(j)}(x_{k+1},y_{k+1})-f^{(j)}(x_{k+1},\bar{y}_{k+1})\|\leq$$

$$\leq L\|y_{k+1}-\bar{y}_{k+1}\|\leq c_4h^{p+1},$$

where $c_4=Lc_3$. $\square$

Remark. According to this result, the method has the global error $p+1$.

3. THE SECOND APPROXIMATION PROCESS

The first approximation process gives us in each meshpoint t_k a set of approximations $\{\bar{y}_k^{(j)},\ j=0,\ldots,p+1\}$ for the values $\{\bar{y}_k^{(j)},\ j=0,\ldots,p+1\}$ of the exact solution in t_k.

We define the spline approximation $s\in C^{p+1}([0,T],\mathbb{R}^m)$ in the following way:

$$s(t):=\overline{H}_{k+1}(t),\quad t\in[t_k,t_{k+1}],\quad k=0,\ldots,N-1. \tag{15}$$

Clearly, s is a deficient spline, of degree $2p+3$ and constant defect $p+2$. The effective construction may be performed in different ways. One of them is by using (11).

Estimates for the global error are given by the following result:

THEOREM 4. *Let* y *be the exact solution of* (1) *and* s *the spline constructed in* (15). *Then there is a positive constant* c_5 *independent of* h *for which the inequalities*

$$\|y^{(j)}(t)-s^{(j)}(t)\|\leq c_5h^{p+1-j},\qquad j=0,\ldots,p+1$$

hold for any $t\in[0,T]$.

Proof. For $x \in [t_k, t_{k+1}]$

$$\|y(t) - s(t)\| \le \|y(t) - H_{k+1}(t)\| + \|H_{k+1}(t) - \overline{H}_{k+1}(t)\| \le$$

$$\le Mh^{p+1} + \left\| \sum_{j=0}^{p+1} g_{kj}(x)(y_k^{(j)} - \overline{y}_k^{(j)}) + \sum_{j=0}^{p+1} g_{k+1j}(x)(y_{k+1}^{(j)} - \overline{y}_{k+1}^{(j)}) \right\| \le$$

$$\le Mh^{p+1} + \|y_k - \overline{y}_k\| h \|g_{k,0}\|_\infty + \|y_k - \overline{y}_k\| Lh \sum_{j=1}^{p+1} \|g_{kj}\|_\infty +$$

$$+ \|y_{k+1} - \overline{y}_{k+1}\| h \|g_{k+1,0}\|_\infty + \|y_{k+1} - \overline{y}_{k+1}\| Lh \sum_{j=1}^{p+1} \|g_{k+1j}\|_\infty \le$$

$$\le Mh^{p+1} + c_4 \|g_{k,0}\|_\infty h^{p+2} + c_4 L \sum_{j=1}^{p+1} \|g_{kj}\|_\infty + h^{p+2} +$$

$$+ c_4 \|g_{k+1,0}\|_\infty h^{p+2} + c_4 L \sum_{j=1}^{p+1} \|g_{k+1j}\|_\infty + h^{p+2} \le c_5 h^{p+1}. \quad \square$$

4. STABILITY

For studying the stability of the method, we apply the first approximation process to the following 1D test problem

(16) $$y' = \lambda y, \quad \lambda \in \mathbb{C}$$
$$y(0) = y_0.$$

The exact solution of (16) is in $C^\infty([0, T])$. So, for the Hermite-interpolator H_{k+1} with two knots and the data up to the $(p+1)$-th derivative, the following identity holds

$$y(x) - H_{k+1}(x) = \frac{y^{(2p+4)}(\zeta_{k+1})}{(2p+4)!}(x - t_k)^{p+2}(x - t_{k+1})^{p+2}, \quad x \in [t_k, t_{k+1}],$$

where C is a constant independent of h. If we follow the argumentation from Lemma 2.1 and Theorem 2.2, we obtain the following global-error estimate for the approximate solution of (16)

$$|y(x) - H_{k+1}(x)| \le Ch^{2p+4}, \; x \in [t_k, t_{k+1}],$$

with c_6 a positive constant independent of h. So, for the test problem (16) the method has the order $2p + 4$.

Applying the method (4)–(12) to (16), we get

$$|y_{k+1} - \overline{y}_{k+1}| \le c_6 h^{2p+4},$$

The stability function is

(17) $$R(\lambda h) = \frac{1 + \sum_{j=0}^{p+1} \int_{t_k}^{t_{k+1}} g_{kj}(x)\, dx \lambda^{j+1}}{1 - \sum_{j=0}^{p+1} \int_{t_k}^{t_{k+1}} g_{k+1j}(x)\, dx \lambda^{j+1}},$$

a rational function for which both the numerator and the denominator are polynomials of degree $p + 2$.

THEOREM 5. *The spline method is A-stable for all* $p \ge 0$.

Proof. The method applied to the test problem (16) has the global order $2p + 4$, i.e.,

$$R(z) = 1 + \frac{z}{1!} + \dots + \frac{z^{2p+4}}{(2p+4)!} + \mathcal{O}(h^{2p+5}), \quad z := \lambda h,$$

so

$$e^z = R(z) + \mathcal{O}(h^{2p+5}).$$

This means that $R(z)$ is the $(p + 2, p + 2)$ Padé-approximation of the exponential. Following [1, Theorem 4.12 p. 60], $R(z)$ is A-acceptable and thus the method is A-stable.$\square$

Remark. As the stability function is a Padé-approximation, we have for $R(z)$ the following formula (see [1, p. 50])

(18) $$R(z) = \frac{1 + \sum_{j=0}^{p+1} \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!} z^j}{1 + \sum_{j=0}^{p+1} (-1)^j \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!} z^j}.$$

If we want to apply the method for stiff linear problems, we may be interested in a better existence result than the one provided by Theorem 1. The following theorem gives an existence condition for linear systems of differential equations.

Consider the system

(19)
$$y'(t) = Ay(t) + b(t)$$
$$y(0) = y_0$$

with $t \in [0, T]$, $A \in M_{m\times m}(\mathbb{R})$, $b \in C^p([0, T], \mathbb{R})$. Denote by $\sigma(A)$ the set of the m eigenvalues of the matrix A.

THEOREM 6. *The discrete implicit method (3)–(5) applied to the system (7) has a unique solution for the step h if and only if*

$$\frac{\xi_k}{h} \notin \sigma(A),$$

where $\xi_k, k = 1, \dots, p+2$ are the roots of the polynomial

(20)
$$P(z) = 1 + \sum_{j=1}^{p+2} \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!} z^j.$$

Proof. If we compare (17) and (18), we observe that

(21)
$$\int_{t_k}^{t_{k+1}} g_{kj-1}(x)\,dx = \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!}, \quad j = 1, \dots, p+2$$

and

(22)
$$\int_{t_k}^{t_{k+1}} g_{k+1j-1}(x)\,dx = (-1)^j \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!}, \quad j = 1, \dots, p+2.$$

Applying the method (4)–(12) to the problem (19), we get after some calculations

$$\bar{y}_{k+1} - A \sum_{j=0}^{p+1} \int_{t_k}^{t_{k+1}} g_{k+1j}(x)\,dx \bar{y}_{k+1}^{(j)} =$$

$$= \bar{y}_k + A \sum_{j=0}^{p+1} \int_{t_k}^{t_{k+1}} g_{kj}(x)\,dx \bar{y}_k^{(j)} + \int_{t_k}^{t_{k+1}} b(x)\,dx$$

and using (21)–(22)

$$\bar{y}_{k+1} + A \sum_{j=1}^{p+2} (-1)^j \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!} h^j \bar{y}_{k+1}^{(j-1)} =$$

$$= \bar{y}_k + A \sum_{j=1}^{p+2} \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!} h^j \bar{y}_{k+1}^{(j-1)} + \int_{t_k}^{t_{k+1}} b(x)\,dx.$$

From (12) and the last equality, we get the $m(p+2) \times m(p+2)$ linear system

$$\begin{bmatrix} \mathbf{I}_m - \frac{h}{2}A & \frac{h^2}{10}A & -\frac{h^3}{120}A & \dots & (-1)^{p+2} h^{p+2} \frac{(p+2)!}{(2p+4)!} A \\ -A & \mathbf{I}_m & \mathbf{O}_m & \dots & \mathbf{O}_m \\ -A^2 & \mathbf{O}_m & \mathbf{I}_m & \dots & \mathbf{O}_m \\ \dots & \dots & \dots & & \dots \\ -A^{p+1} & \mathbf{O}_m & \mathbf{O}_m & \dots & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \bar{y}_{k+1} \\ \bar{y}'_{k+1} \\ \bar{y}''_{k+1} \\ \dots \\ \bar{y}_{k+1}^{(p+1)} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \dots \\ d_{p+2} \end{bmatrix}$$

where

$$d_1 = \left(\mathbf{I}_m + \sum_{j=1}^{p+2} \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!} h^j A^j \right) \bar{y}_k + \int_{t_k}^{t_{k+1}} b(x)\,dx$$

$$d_{i+1} = \sum_{j=0} A^{i-j} b^{(j)}(t_{k+1}), \quad i = 0, \dots, p+1.$$

By substitutions, we get the $m \times n$ linear system

(23)
$$\left(\mathbf{I}_m + \sum_{j=1}^{p+2} (-1)^j \frac{(p+2)!}{(2p+4)!} \frac{(2p+4-j)!}{j!(p+2-j)!} h^j A^j \right) \bar{y}_{k+1} = \mathbf{d},$$

where $\mathbf{d} \in \mathbb{R}^m$ is a free term, independent of $\bar{y}_{k+1}^{(j)}$, $j = 0, \dots, p+1$.

Consider $\{\xi_k, k = 1, \dots, p+2\}$, the roots of the polynomial P given by (20). The matrix of the system (23) can be written as

$$\prod_{k=1}^{p+2} (hA - \xi_k \mathbf{I}_m),$$

so the existence and uniqueness condition for the solution of (23) is

$$\det \left[\prod_{k=1}^{p+2} (hA - \xi_k \mathbf{I}_m) \right] \neq 0$$

or, equivalently,

$$\prod_{k=1}^{p+2} \det (hA - \xi_k \mathbf{I}_m) \neq 0,$$

which proves the conclusion of the theorem. □

Remark. The previous theorem proves that, except for $m(p + 2)$ values of the step h, the method applied to the linear system (19) has a unique solution for every step. This property and the A-stability lead us to the conclusion that the spline method proposed may be suited for numerically solving stiff systems of linear differential equations.

5. FINAL REMARKS AND CONCLUSIONS

In the following we give an interesting interpretation of this spline method. Considering (4), (12) and (15), we can rewrite them in the following way:

$$s(t_{k+1}) = s(t_k) + \int_{t_k}^{t_{k+1}} f(x, s(x))\, dx \tag{24}$$

$$s^{(j+1)}(t_k) = f^{(j)}(t_k, s(t_k)), \quad j = 0, \dots, p \tag{25}$$

$$s^{(j+1)}(t_{k+1}) = f^{(j)}(t_{k+1}, s(t_{k+1})), \quad j = 0, \dots, p. \tag{26}$$

Relations (25) and (26) describe the two-point multiderivative collocation method whereas relation (24) is the collocation on the meshpoints for the equivalent integral equation (3).

In conclusion, the spline method described in this paper is a two-point multiderivative collocation for (1), completed with collocation on the meshpoints for the equivalent integral formulation of (1).

REFERENCES

1. E. Hairer, and G. Wanner, *Solving Ordinary Differential Equations*. II. Stiff and Differential-algebraic Problems, Springer Verlag, 1991.
2. Gh. Micula, *Funcţii spline. Aplicaţii*, Edi. Tehnică, Bucharest, 1978.
3. H. N. Multhey, *A-stabile Kollokationsverfahren mit mehrfachen Knoten*, Computing **29** (1982), 51–61.
4. A. Revnic, *An A-stable spline numerical method for systems of ODEs*, Quaderni Matematici, Università di Trieste *405* (1997).
5. A. Revnic, *An implicit spline-interpolatory numerical method for ODEs*, Studia Univ., Babeş-Bolyai, Mathematica (to appear)

Received November 20, 1997.

Institutul de Calcul "Tiberiu Popoviciu",
C.P.68-1, 3400 Cluj-Napoca,
Romania
E-mail: arevnic@math.ubbcluj.ro