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A DENSITY PROPERTY OF THE C^*-ALGEBRA $PAP(\mathbb{R})$ AND ITS APPLICATIONS

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1. INTRODUCTION

Let $\mathscr{C}(\mathbb{R})$ be the C^*-algebra of bounded continuous complex-valued functions on $\mathbb{R}$, with the supremum norm. Denote by m the Lebesgue measure on $\mathbb{R}$. For $f \in C(\mathbb{R})$ and $a \in \mathbb{R}$, the translate of f by a is the function $R_a f(x) = f(x+a)$, $x \in \mathbb{R}$. A subset $\mathscr{F}$ of $\mathscr{C}(\mathbb{R})$ is said to be translation invariant if $R_a \mathscr{F} \subset \mathscr{F}$, for all $a \in \mathbb{R}$. Throughout this paper, $\|\cdot\|$ denotes the supremum norm on $\mathscr{C}(\mathbb{R})$.

DEFINITION 1.1 [4]. A function $g \in \mathscr{F}(\mathbb{R})$ is called *an almost periodic function on* $\mathbb{R}$, if for each $\varepsilon > 0$, there exists an $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the property that

$$\|R_\tau g - g\| < \varepsilon.$$

Denote by $AP(\mathbb{R})$ the set of all such functions.

Remark 1.1. The set $AP(\mathbb{R})$ is a translation invariant C^*-subalgebra of $\mathscr{C}(\mathbb{R})$ containing the constant functions.

Set

$$PAP_0(\mathbb{R}) = \left\{ \varphi \in \mathscr{C}(\mathbb{R}) : \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} |\varphi(x)|\,dx = 0 \right\}.$$

Remark 1.2. The set $PAP_0(\mathbb{R})$ is a translation invariant C^*-subalgebra of $\mathscr{C}(\mathbb{R})$.

DEFINITION 1.2 [4]. A function $f \in \mathscr{C}(\mathbb{R})$ is called *a pseudo almost periodic function on* $\mathbb{R}$ if $f = g + \varphi$, where $g \in AP(\mathbb{R})$ and $\varphi \in PAP_0(\mathbb{R})$.

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Denote by $PAP(\mathbb{R})$ the set of all pseudo almost periodic functions on $\mathbb{R}$.

THEOREM 1.1 [4]. *$PAP(\mathbb{R})$ is a translation invariant C*-subalgebra of $\mathscr{C}(\mathbb{R})$ containing the constant functions. Furthermore,*

$$PAP(\mathbb{R}) = AP(\mathbb{R}) \oplus PAP_0(\mathbb{R}).$$

DEFINITION 1.3 [4]. A closed subset C of $\mathbb{R}$ is said to be *an ergodic zero set in* $\mathbb{R}$ *if*

$$\lim_{t\to\infty} \frac{m(C \cap [-t,t])}{2t} = 0.$$

THEOREM 1.2 [4]. *Let φ be a function in $\mathscr{C}(\mathbb{R})$. Then $\varphi \in PAP_0(\mathbb{R})$ if and only if, for every $\varepsilon > 0$, the set $C_\varepsilon = \{x \in \mathbb{R} : |\varphi(x)| \geq \varepsilon\}$ is an ergodic zero subset in $\mathbb{R}$.*

Remark 1.3. If $f \in PAP(\mathbb{R})$, then the limit

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(x)\,\mathrm{d}x$$

exists and is finite.

DEFINITION 1.4 [3]. For $f \in PAP(\mathbb{R})$, we shall call *the mean value of the function f*, and we shall denote it by $M(f)$, the limit

$$M(f) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(x)\,\mathrm{d}x.$$

2. A DENSITY PROPERTY OF THE C*-ALGEBRA $PAP(\mathbb{R})$

PROPOSITION 2.1. a) *A closed subset of an ergodic zero set in $\mathbb{R}$ is an ergodic zero set.*

b) *The intersection of the sets of an arbitrary family of ergodic zero sets is an ergodic zero set.*

c) *The union of the sets of a finite family of ergodic zero sets is an ergodic zero set.*

d) *The translate of an ergodic zero set is an ergodic zero set.*

Proof. The first three assertions immediately follow from the definition of an ergodic zero set. We shall prove d).

Let C be an ergodic zero set in $\mathbb{R}$ and let a be a real number. Without loss of generality, we may assume that the number a is in $(0, \infty)$. Let t be a positive, sufficiently large number, such that $a < 2t$. According to the properties of the Lebesgue measure, we obtain

$$\frac{m((a+C)\cap[-t,t])}{2t} = \frac{m(C\cap[-t-a,\ t-a])}{2t} =$$

$$\frac{m((C\cap[-t-a,t-a])\cap[-t,t])}{2t} + \frac{m((C\cap[-t-a,\ t-a])\setminus[-t,t])}{2t} \leq$$

$$\frac{m(C\cap[-t,\ t])}{2t} + \frac{m([-t-a,\ t-a]\setminus[-t,t])}{2t} =$$

$$= \frac{m(C\cap[-t,t])}{2t} + \frac{a}{2t}.$$

Since

$$\lim_{t\to\infty} \frac{m(C\cap[-t,t])}{2t} = 0,$$

it follows

$$\lim_{t\to\infty} \frac{m((a+C)\cap[-t,t])}{2t} = 0.$$

Q.E.D.

Remark 2.1. It follows from Theorem 1.2 that $PAP_0(\mathbb{R})$ is the set of all bounded continuous complex-valued functions f on $\mathbb{R}$, such that for every $\varepsilon > 0$, there exists an ergodic zero set C in $\mathbb{R}$ (depending on f and ε) such that $|f(x)| < \varepsilon$ for all $x \in \mathbb{R}\setminus C$.

Let $\mathscr{C}_{0-erg}(\mathbb{R})$ be the set of all bounded continuous complex-valued functions f on $\mathbb{R}$ such that there exists an ergodic zero set C in $\mathbb{R}$ (depending on f) such that $f(x) = 0$ for all $x \in \mathbb{R}\setminus C$. In other words, $\mathscr{C}_{0-erg}(\mathbb{R})$ is the set of all bounded continuous functions on $\mathbb{R}$ with ergodic zero support.

THEOREM 2.1. a) *$\mathscr{C}_{0-erg}(\mathbb{R})$ is a subalgebra of the algebra $PAP_0(\mathbb{R})$.*

b) *$\mathscr{C}_{0-erg}(\mathbb{R})$ is uniformly dense in $PAP_0(\mathbb{R})$.*

Proof. a) Our claim follows from the properties of the support of a function and from Proposition 2.1. We shall prove b).

Let f be a function in $PAP_0(\mathbb{R})$ and let ε be a positive number. Thus, the set $C_\varepsilon = \left\{x \in \mathbb{R} : |f(x)| \geq \frac{\varepsilon}{2}\right\}$ is an ergodic zero set. It follows

$$\lim_{t\to\infty} \frac{m(C_\varepsilon \cap [-t,t])}{2t} = 0$$

and, therefore,

$$\lim_{t\to\infty}\frac{m((\mathbb{R}\setminus C_\varepsilon)\cap[-t,t])}{2t}=1.$$

Taking into account that $\mathbb{R}\setminus C_\varepsilon$ is an open set, hence Lebesgue measurable, we can find a closed subset $F_\varepsilon\subset\mathbb{R}\setminus C_\varepsilon$ such that

$$m((\mathbb{R}\setminus C_\varepsilon)\setminus F_\varepsilon)<1.$$

Therefore

$$\lim_{t\to\infty}\frac{m(((\mathbb{R}\setminus C_\varepsilon)\setminus F_\varepsilon)\cap[-t,t])}{2t}=0.$$

Clearly, we have

$$\frac{m((\mathbb{R}\setminus C_\varepsilon)\cap[-t,t])}{2t}=\frac{m(((\mathbb{R}\setminus C_\varepsilon)\setminus F_\varepsilon)\cap[-t,t])}{2t}+\frac{m(F_\varepsilon\cap[-t,t])}{2t}.$$

Combining the above equality with (1) and (2), we obtain

$$\lim_{t\to\infty}\frac{m(F_\varepsilon\cap[-t,t])}{2t}=1.$$

Making use of the fact that F_ε and C_ε are disjoint closed sets in $\mathbb{R}$, which is a T_4 space, we find an open set D_ε such that

$$F_\varepsilon\subset D_\varepsilon\subset\overline{D}_\varepsilon\subset\mathbb{R}\setminus C_\varepsilon.$$

It follows from the inclusions above and from the equalities

$$\lim_{t\to\infty}\frac{m(F_\varepsilon\cap[-t,t])}{2t}=\lim_{t\to\infty}\frac{m((\mathbb{R}\setminus C_\varepsilon)\cap[-t,t])}{2t}=1$$

that

$$\lim_{t\to\infty}\frac{m(D_\varepsilon\cap[-t,t])}{2t}=\lim_{t\to\infty}\frac{m(\overline{D}_\varepsilon\cap[-t,t])}{2t}=1.$$

Furthermore,

$$\lim_{t\to\infty}\frac{m((\mathbb{R}\setminus D_\varepsilon)\cap[-t,t])}{2t}=0,$$

hence $\mathbb{R}\setminus D_\varepsilon$ is an ergodic zero set.

Now we use the fact that $\mathbb{R}$ is a normal space and applying Uryson's theorem, we find that there exists a continuous function $g, g:\mathbb{R}\to[0,1]$, such that $g(x)=1$ for all $x\in C_\varepsilon$ and $g(x)=0$ for all $x\in\overline{D}_\varepsilon$.

Afterwards, we consider the continuous function $F:\mathbb{R}\to C$ defined by $F(x)=f(x)\cdot g(x)$ for all $x\in\mathbb{R}$. It is obvious that $F(x)=f(x)$ for all $x\in C_\varepsilon$ and $F(x)=0$ for all $x\in\overline{D}_\varepsilon$. We obtain that $F(x)=0$ for all $x\in D_\varepsilon=\mathbb{R}\setminus(\mathbb{R}\setminus D_\varepsilon)$ and $\mathbb{R}\setminus D_\varepsilon$ is an ergodic zero set. This means that $F\in\mathscr{C}_{0\text{-}erg}(\mathbb{R})$.

On the other hand, we have that for all $x\in\mathbb{R}$, $|F(x)|\le|f(x)|$ and from this inequality it follows that for all $x\in\mathbb{R}$

$$|f(x)-F(x)|<\varepsilon.\quad \text{Q.E.D.}$$

Denote by $\hat{\mathbb{R}}$ the group of the characters of the group $\mathbb{R}$ and by $\langle\hat{\mathbb{R}}\rangle$ the subspace generated by the set $\hat{\mathbb{R}}$ in the Banach space $\mathscr{C}(\mathbb{R})$.

COROLLARY 2.1. *$PAP(\mathbb{R})$ is the smallest C^*-subalgebra of the C^*-algebra $\mathscr{C}(\mathbb{R})$ containing the characters of the group $\mathbb{R}$ and the functions of the algebra $\mathscr{C}_{0\text{-}erg}(\mathbb{R})$.*

Proof. Let A be a C^*-subalgebra of the C^*-algebra $\mathscr{C}(\mathbb{R})$, containing the characters of the group $\mathbb{R}$ and the functions of the algebra $\mathscr{C}_{0\text{-}erg}(\mathbb{R})$. It is known that

$$\overline{\langle\hat{\mathbb{R}}\rangle}=AP(\mathbb{R})$$

in the sense of uniform convergence on $\mathbb{R}$ [1]; combining this fact with the hypothesis that A is a Banach algebra, we obtain the inclusion $AP(\mathbb{R})\subset A$. Now, applying Theorem 2.1, it follows that

$$\overline{\mathscr{C}_{0\text{-}erg}(\mathbb{R})}=PAP_0(\mathbb{R})$$

in the sense of uniform convergence on $\mathbb{R}$ and by virtue of the same property of the algebra A we see that $PAP_0(\mathbb{R})\subset A$. So, $PAP(\mathbb{R})\subset A$. Q.E.D.

Remark 2.2. Observe that

$$PAP(\mathbb{R})=\overline{\langle\hat{\mathbb{R}}\rangle}\oplus\overline{\mathscr{C}_{0\text{-}erg}(\mathbb{R})}$$

in the sense of uniform convergence on $\mathbb{R}$.

3. SOME APPLICATIONS OF THE DENSITY PROPERTY OF THE C^*-ALGEBRA $PAP(\mathbb{R})$

PROPOSITION 3.1. *Let $\mu: PAP(\mathbb{R})\to C$ be a linear functional such that:*

a) $\mu(f)\ge 0$ *for all* $f\in PAP(\mathbb{R})$, $f\ge 0$;

b) $\mu(f)=1$ *for* $f\equiv 1$;

c) $\mu(\gamma)=\mu(R_a\gamma)$ *for all* $a\in\mathbb{R}$ *and for all* $\gamma\in\hat{\mathbb{R}}$;

d) $\mu(\varphi)=0$ *for all* $\varphi\in\mathscr{C}_{0\text{-}erg}(\mathbb{R})$.

Then $\mu \equiv M$ *(M is the mean value defined on* $PAP(\mathbb{R})$*).*

Proof. For every $f \in PAP(\mathbb{R})$ we obtain that

$$|\mu(f)| \leq \|f\|.$$

It follows that μ is a continuous functional on $PAP(\mathbb{R})$.

Set

$$A = \{f \in PAP(\mathbb{R}) : \mu(f) = \mu(R_a f) \text{ for all } a \in \mathbb{R}\}.$$

We can easily see that A is a closed subspace of the Banach space $PAP(\mathbb{R})$. From the hypothesis we have $\hat{\mathbb{R}} \subset A$ and, therefore, $\overline{\langle \hat{\mathbb{R}} \rangle} = AP(\mathbb{R}) \subset A$. (The closure $\overline{\langle \hat{\mathbb{R}} \rangle}$ of $\langle \hat{\mathbb{R}} \rangle$ is considered in the sense of uniform convergence on $\mathbb{R}$). Applying Theorem 18.9 in [2], we obtain the equality $\mu \equiv M$ on $AP(\mathbb{R})$.

Let φ be a function in $PAP_0(\mathbb{R})$. By Theorem 2.1, we find a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of functions of the algebra $\mathscr{C}_{0-erg}(\mathbb{R})$ such that $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$. Clearly, we have

$$\mu(\varphi) = \mu(\lim_{n\to\infty} \varphi_n) = \lim_{n\to\infty} \mu(\varphi_n) = 0.$$

Thus $\mu \equiv 0$ on $PAP_0(\mathbb{R})$, and, furthermore, $\mu \equiv M$ on $PAP(\mathbb{R})$. Q.E.D.

PROPOSITION 3.2. *Let* $(T_n)_{n \in \mathbb{N}}$ *be a sequence of positive linear operators,* $T_n : PAP(\mathbb{R}) \to PAP(\mathbb{R})$ *for all* $n \in \mathbb{N}$, *such that:*

a) $\lim_{n\to\infty} T_n f = f$ *uniformly for each* $f \in \hat{\mathbb{R}}$ *and for each* $f \in \mathscr{C}_{0-erg}(\mathbb{R})$;

b) $T_n f = f$ *for* $f \equiv 1$, *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty} T_n f = f$ *uniformly for each* $f \in PAP(\mathbb{R})$.

Proof. Set

$$A = \{f \in PAP(\mathbb{R}) : \lim_{n\to\infty} \|T_n f - f\| = 0\}.$$

We shall prove that A is a closed subspace of the Banach space $PAP(\mathbb{R})$.

Consider a sequence $(f_k)_{k \in \mathbb{N}}$ of functions of A such that

$$\lim_{n\to\infty} \|f_k - f_0\| = 0,$$

where $f_0 \in PAP(\mathbb{R})$.

If $T : PAP(\mathbb{R}) \to PAP(\mathbb{R})$ is a positive linear operator, then the following inequality holds for each $f \in PAP(\mathbb{R})$

$$\|Tf\| \leq \|f\| \cdot \|T1\|.$$

It follows for all $n, k \in \mathbb{N}$ and for all $f \in PAP(\mathbb{R})$

$$\|T_n f - f\| \leq \|T_n f - T_n f_k\| + \|T_n f_k - f_k\| + $$
$$+ \|f_k - f\| \leq \|T_n f_k - f_k\| + 2\|f_k - f\|.$$

Let ε be a positive number.

By $\lim_{k\to\infty} \|f_k - f_0\| = 0$ we have that there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0, \|f_k - f_0\| < \frac{\varepsilon}{3}$. Now, using the fact that $\lim_{n\to\infty} \|T_n f_{k_0} - f_{k_0}\| = 0$, we find a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|T_n f_{k_0} - f_{k_0}\| < \frac{\varepsilon}{3}$.

It follows from (3) that for all $n \geq n_0$

$$\|T_n f_0 - f_0\| < \varepsilon.$$

Hence, $\lim_{n\to\infty} \|T_n f_0 - f_0\| = 0$. This means that $f_0 \in A$, so A is a closed subspace of the Banach space $PAP(\mathbb{R})$.

On the other hand we have that

$$PAP(\mathbb{R}) = \overline{\langle \hat{\mathbb{R}} \rangle \oplus \mathscr{C}_{0-erg}(\mathbb{R})}$$

and, taking into account that $\hat{\mathbb{R}} \subset A$ and $\mathscr{C}_{0-erg}(\mathbb{R}) \subset A$, we obtain $A = PAP(\mathbb{R})$. Q.E.D.

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