COMPUTING RELIABLE ERROR ESTIMATIONS
FOR INTEGRAL EQUATIONS WITH DISCONTINUITIES

HANS-JÜRGEN DOBNER

1. INTRODUCTION

We are concerned with numerical methods providing a software-assisted error treatment (briefly called enclosure methods) for Fredholm integral equations

$$(1.1) \qquad x(s) = g(s) + \int_\alpha^\beta k(s,t)\, x(t)\, dt, \quad \alpha \le s \le \beta;$$

$I := [\alpha, \beta]$ is a finite real interval, g real-valued and continuous. Throughout this paper we assume that the kernel has the following properties:

(P1) $k(s,t)$ is Lebesgue integrable with respect to t for each $s \in [\alpha, \beta]$ and

$$\|k\| = \sup_{\alpha \le s \le \beta} \int_\alpha^\beta |k(s,t)|\, dt < \infty;$$

(P2) $\displaystyle \lim_{|s_1 - s_2| \to 0} \int_\alpha^\beta |k(s_1, t) - k(s_2, t)|\, dt = 0$ uniformly for $\alpha \le s_1, s_2 \le \beta$.

This includes, e.g., discontinuous derivatives in the kernel or weakly singular kernels.

Therefore, the operator k (using the same notation for the kernel as well as for the corresponding operator)

$$(1.2) \qquad k(x)(s) = \int_\alpha^\beta k(s,t)\, x(t)\, dt, \quad \alpha \le s \le \beta,$$

is a compact operator on $C[\alpha, \beta]$ into $C[\alpha, \beta]$. $C[\alpha, \beta]$ equipped with the maximum norm $\|\cdot\|_\infty$ is a Banach space.

For deriving close bounds, a calculus with the following features must be feasible (see Adams and Kulisch [1]): for taking into account round-off errors, a

---

precise computer and interval arithmetic, and for estimating approximation errors, enclosure tools for basic problems in numeric (e.g. integration, evaluation of expressions, determination of ranges, Taylor and Fourier expansions, linear and nonlinear equations).

We denote the set of closed real intervals by $I\mathbb{R}$, and the set of interval vectors and matrices by $I\mathbb{R}^n$ and $I\mathbb{R}^{n \times m}$.

The elementary arithmetic operations $\mathscr{O} = \{+, -, \cdot, /\}$ for interval objects are defined in terms of their endpoints (see Moore [10]). For $A = [\underline{A}, \overline{A}] \in I\mathbb{R}$, diameter and absolute value are defined as

$$\mathrm{diam}(A) := \overline{A} - \underline{A}, \quad |A| := \max\{|a|; \ a \in A\}.$$

For intervals $A, B, C, D \in I\mathbb{R}$ we list some important rules:
– inclusion monotonicity

$$(1.3) \qquad A \subseteq C, \quad B \subseteq D \Rightarrow A \circ B \subseteq C \circ D, \quad o \in \mathscr{O};$$

– blowing-up phenomena

$$(1.4) \qquad \mathrm{diam}(A \pm B) = \mathrm{diam}(A) + \mathrm{diam}(B);$$

– subdistributive property

$$(1.5) \qquad A(B+C) \subseteq AB + AC.$$

Some special cases in which distributivity does hold are, e.g.,

$$(1.6) \qquad a \in \mathbb{R} \Rightarrow a(B+C) = aB + aC,$$

$$(1.7) \qquad BC > 0 \Rightarrow A(B+C) = AB + AC.$$

(For a complete discussion see Ratschek [12].) An interval-valued function F is said to be an interval extension of a real-valued function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ iff

$$(1.8) \qquad f(t) \in F(t) \quad \text{for} \quad t \in D.$$

The set of interval extensions of f is abbreviated to $\mathscr{I}(f)$. It is important to note that F contains all functions whose graph is lying in F.

Advices for constructing such interval extensions are given in Dobner [5], Hammer et al. [7] and Klein [8]. If an interval T is substituted for the real variable t, all operations occurring in f are interpreted as interval operations, hence $f(T) \in I\mathbb{R}$. For a detailed description concerning interval arithmetic or topological properties of $I\mathbb{R}$ see Moore [10].

In the next sections decomposition techniques are considered. The last paragraph contains numerical examples and computational notes. It starts by converting the integral equation into Riemann sums by use of mean value principles.

2. A RIEMANN-LIKE APPROACH

This completely new method of solving Fredholm integral equations numerically resembles the definition of the integral by Riemann sums. In order to apply this method, it is necessary that x have no zeros in $[\alpha, \beta]$. If an upper bound b for $|x|$ is known, then

$$(2.1) \qquad y(s) = g(s) - b \int_{\alpha}^{\beta} k(s,t)\,dt + b + \int_{\alpha}^{\beta} k(s,t)\,y(t)\,dt, \quad \alpha \le s \le \beta,$$

is an equation with a nonnegative solution $y(s) = x(s) + b, \alpha \le s \le \beta$.

THEOREM 2.1. *We assume $x(s) \ne 0, \alpha \le s \le \beta$, and divide the interval $[\alpha, \beta]$ into n subintervals $I_l = [\alpha_l, \beta_l]$ of length $h_l, l = 1, \dots, n$. Let $K \in \mathscr{I}(k), G \in \mathscr{I}(g)$ and let $U = (U_1, \dots, U_n) \in I\mathbb{R}^n$ satisfy the linear system*

$$(2.2) \qquad \sum_{l=1}^{n} (\delta_{jl} - h_l K(I_j, I_l)) U_l = G(I_j), \quad j = 1, \dots, n.$$

Then the solution x of (1.1) exists within X, more precisely

$$(2.3) \qquad x(s) \in X(s) := G(s) + \sum_{l=1}^{n} h_l K(s, I_l) U_l, \quad \text{for} \quad s \in I.$$

Proof. Since $x(s) \ne 0$, we have for $s \in I$

$$x(s) = g(s) + \int_{\alpha}^{\beta} k(s,t)\,x(t)\,dt = g(s) + \sum_{l=1}^{n} \int_{\alpha_l}^{\beta_l} k(s,t)\,x(t)\,dt =$$

$$= g(s) + \sum_{l=1}^{n} k(s, \eta_l) \int_{\alpha_l}^{\beta_l} x(t)\,dt = g(s) + \sum_{l=1}^{n} k(s, \eta_l)\,x(\tau_l)\,h_l$$

with intermediate values $\eta_l, \tau_l \in I_l, l = 1, \dots, n$. The τ_l are independent of s, therefore we can set $s = \tau_j, j = 1, \dots, n$, to obtain the system

$$(2.4) \qquad \sum_{l=1}^{n} (\delta_{jl} - h_l k(\tau_j, \eta_l))\,x(\tau_l) = g(\tau_j), \quad j = 1, \dots, n.$$

Since $\eta_l, \tau_l \in I_l, l = 1, \dots, n$, we substitute in (2.4) the interval quantities $I_1, \dots, I_n$ for the intermediate values, thus yielding the interval system

$$(2.5) \qquad \sum_{l=1}^{n} (\delta_{jl} - h_l k(I_j, I_l))\,x(I_l) = g(I_j), \quad j = 1, \dots, n;$$

the assertion now follows by the properties of K and G together with interval analytical arguments. □

In [4], methods for computing enclosures of type (2.2) are discussed.

Remark 2.1. From (2.2) the error estimate

$$(2.6) \qquad |x(s) - X(s)| \le \frac{1}{2} \operatorname{diam}(X(s)), \quad s \in I,$$

is derived and the convergence of the enclosure solution X to the true solution x as $h \to 0$, $h := \max_{j=1,\dots,n} \{h_j\}$, is demonstrated.

Remark 2.2. The nonsingularity of the matrix set occurring in (2.2) is checked during the computational process by an enclosure method for linear systems (cf. Hammer et al. [7]).

3. DECOMPOSITION OF THE KERNEL

This approach is favourable when a Fourier expansion for the kernel is given. If $k \in L^2([\alpha, \beta])$ and φ_i is a complete orthonormal system, then

$$(3.1) \qquad k(s,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} \varphi_i(s) \varphi_j(t),$$

where

$$(3.2) \qquad c_{ij} = \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} k(s,t) \varphi_i(s) \varphi_j(t) \, ds \, dt, \quad i, j = 1, 2, \dots.$$

We approximate k through a degenerate kernel k_d

$$(3.3) \qquad k_d = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \varphi_i(s) \varphi_j(t) = \sum_{i=1}^{n} \varphi_i(s) \sum_{j=1}^{n} c_{ij} \varphi_i(t)$$

$$(3.4) \qquad k_c = k - k_d.$$

We suppose that the Fourier coefficients $c_{ij}, i, j = 1, \dots, n$, are either given analytically or can be computed with tight bounds (cf. Storck [13]), so that intervals c_{ij} with $c_{ij} \in \mathcal{J}(c_{ij})$, $i, j = 1, \dots, n$, are available; thus

$$(3.5) \qquad \|k_c\|_2^2 = \|k\|_2^2 - \|k_d\|_2^2 =$$

$$= \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} |k(s,t)|^2 \, ds \, dt - \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}|^2$$

is a computable estimate for $\|k_c\|_2$; the integral can be evaluated analytically in many cases. The number n is chosen such that

$$(3.6) \qquad \|k_c\|_2 < 1.$$

We define

$$(3.7) \qquad a_i(s) := \varphi_i(s), \quad A_i \in \mathcal{J}(a_i), \quad i = 1, \dots, n,$$

$$(3.8) \qquad b_i(s) := \sum_{j=1}^{n} c_{ij} \varphi_i(s), \quad B_i \in \mathcal{J}(b_i), \quad i = 1, \dots, n.$$

THEOREM 3.1. *Let k_c in (3.4) be such that (3.6) is true. If, furthermore, $V \in \mathcal{J}(v)$, $W_i \in \mathcal{J}(w_i)$, $i = 1, \dots, n$, where*

$$(3.9) \qquad v(s) = g(s) + \int_{\alpha}^{\beta} k_c(s,t) v(t) \, dt,$$

$$(3.10) \qquad w_i(s) = a_i(s) + \int_{\alpha}^{\beta} k_c(s,t) w_i(t) \, dt, \quad i = 1, \dots, n,$$

and $U = (U_1, \dots, U_n) \in \mathcal{J}(u)$, where $u = (u_1, \dots, u_n)$ is a solution of

$$(3.11) \qquad \sum_{i=1}^{n} \left(\delta_{ij} - \int_{\alpha}^{\beta} b_j(t) w_i(t) \, dt \right) u_i = \int_{\alpha}^{\beta} b_j(t) v(t) \, dt, \quad j = 1, \dots, n,$$

then there exists a continuous solution of (1.1) satisfying

$$(3.12) \qquad x(s) \in X(s) = V(s) + \sum_{i=1}^{n} U_i W_i(s), \quad \alpha \le s \le \beta.$$

Proof. We have

$$x(s) = v(s) + \sum_{i=1}^{n} u_i w_i(s), \quad \alpha \le s \le \beta,$$

and the enclosure properties of $V, U = (U_1, \dots, U_n)$ and $A_i, i = 1, \dots, n$, imply (3.12). □

The extensions required for (3.9) and (3.10) are obtained iteratively by using a modification of Schauder's fixed point theorem (see Dobner [4]).

4. DECOMPOSITION OF THE DOMAIN

We consider now special kernels

$$k(s,t) = p(|s-t|) q(s,t),$$

where q represents the well-behaved, sufficiently smooth part, and p is known to satisfy $(P1)$ and $(P2)$.

The idea is to transform the weakly singular equation into a system of equations. Such a decomposition has some numerical advantages concerning accuracy and computation time (see Klein [8]). We subdivide the domain

$$\alpha_i := \alpha + \frac{i-1}{N}(\beta - \alpha), \quad \beta_i = \alpha_{i+1}, \quad i = 1, \ldots, N,$$

and define with $i, j = 1, \ldots, N$

(4.1)
$$g_i(s) := \begin{cases} g(s), & s \in [\alpha_i, \beta_i], \\ 0, & \text{else} \end{cases}$$

(4.2)
$$k_{ij}(s) := \begin{cases} k(s,t), & (s,t) \in [\alpha_i, \beta_i] \times [\alpha_j, \beta_j], \\ 0, & \text{else} \end{cases}$$

(4.3)
$$y_i(s) := \begin{cases} x(s), & s \in [\alpha_i, \beta_i], \\ 0, & \text{else} \end{cases}$$

thus leading to the system

(4.4)
$$y_i(s) = g_i(s) + \sum_{j=1}^{N} \int_{\alpha_j}^{\beta_j} k_{ij}(s,t)\, y_j(t)\, dt, \quad i = 1, \ldots, N.$$

THEOREM 4.1. *Equation (1.1) is equivalent to the system (4.4) in the sense that any solution of (1.1) is also a solution of (4.4) and vice versa.*

Proof. Cf. Michlin [9]. $\square$

Each kernel k_{ij} is written as a sum of a degenerate part k_{ij}^d and a contractive one k_{ij}^c (cf. §3):

(4.5)
$$k_{ij} = k_{ij}^d + k_{ij}^c, \quad i, j = 1, \ldots, N.$$

This splitting is performed

(4.6)
$$\begin{cases} k_{ij}^d = 0 \\ k_{ij}^c = k_{ij} \end{cases}, \quad \text{if } |i-j| < 2, \; i, j = 1, \ldots, N,$$

(4.7)
$$\begin{cases} k_{ij}^d = \sum_{v=1}^{n} a_{ij}^{(v)}(s)\, b_{ij}^{(v)}(t) \\ k_{ij}^c = k_{ij} - k_{ij}^d \end{cases}, \quad \text{if } |i-j| \geq 2, \; i, j = 1, \ldots, N,$$

in such a way that

(4.8)
$$\|K^c\| := \max_{i=1}^{N} \left\{ \sum_{j=1}^{n} \|k_{ij}^c\| \right\} = \max_{i=1}^{N} \left\{ \sum_{j=1}^{n} \max_{s \in [\alpha_i, \beta_i]} \int_{\alpha_j}^{\beta_j} |k_{ij}^c(s,t)|\, dt \right\} < 1,$$

$$k^c = (k_{ij}^c)_{i,j=1,\ldots,N} \in (K_{ij}^c)_{i,j=1,\ldots,N} = K^c.$$

Bounds for the solution of (4.4) are obtained with the method derived in Section 3, where scalar quantities have to be replaced by corresponding vector quantities. The result is summarized in

THEOREM 4.2. *Let $Y = (Y_1, \ldots, Y_N) \in \mathfrak{I}(y) = \mathfrak{I}(y_1, \ldots, y_N)$. Then the solution x of the weakly singular equation is piecewise enclosed in*

$$x(s) \in Y_i(s), \quad s \in [\alpha_i, \beta_i], \quad i = 1, \ldots, N.$$

Proof. It follows from Theorem 4.1 together with the properties of Y. $\square$

5. NUMERICAL EXAMPLES AND COMPUTATIONAL NOTES

In this paper we have constructed enclosure methods for Fredholm integral equations of the second kind. The overriding concern of this algorithm is reliability – the verification of the result is established automatically.

In contrast to classical numerical schemes, where the errors arising during computation remain unknown, enclosure methods provide an approximation together with mathematical guaranteed estimations for all kinds of numerical errors (including round-off and approximation errors). In order to illustrate the fundamental difference between these two principles, we consider the equation

$$x(s) = e^{-s} - \frac{1}{2} + \frac{1}{2}e^{-(s+1)} + \frac{1}{2}\int_0^1 (s+1)\, e^{-st} x(t)\, dt, \quad 0 \leq s \leq 1,$$

and compare their computed results:

Quadrature formula $n = 16$	Enclosure method (here kernel decomposition with $n = 10$)
$x\left(\frac{1}{2}\right) = 0.607208659$	$x\left(\frac{1}{2}\right) \in 0.606530661 + [-1,1] \cdot 1.505485\, e^{-8}$

We shall apply the enclosure schemes to problems which cover the most important types of kernels (C^∞-kernels, unbounded kernels, discontinuous kernels).

The computations were done on a personal computer with an MC 68000 processor. The programs were written in PASCAL-XSC with an optimal decimal arithmetic of 13 digits.

For the Riemann method a division into equidistant subintervals was chosen. In the column "accuracy" of the following tables we display the accuracy of the computed solution which has been achieved in the mean.

C^{∞}-kernel

$$x(s) = 1 - \frac{0.9090909}{2\pi} \int_0^{2\pi} \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(s+t)} x(t)\, dt, \quad 0 \le s \le 2\pi.$$

Riemann approach		Kernel decomposition	
n	accuracy	n	accuracy
40	$2.876 \cdot 10^{-2}$	9	$7.096 \cdot 10^{-5}$
80	$1.431 \cdot 10^{-2}$	27	$7.207 \cdot 10^{-14}$

We begin with a C^{∞}-kernel to demonstrate the great advantage of the kernel decomposition concerning accuracy and execution time for kernel functions with series expansions. Here the Fourier series of k was decomposed as described in Section 3. Of course, these excellent results cannot be obtained by the Riemann method, where we observe a linear behaviour of the error.

Unbounded kernel

$$x(s) = s - \frac{\sqrt{s^3}}{75} - \frac{2s+1}{150}\sqrt{1-s} + \frac{1}{100}\int_0^1 \frac{x(t)}{\sqrt{|s-t|}}\, dt, \quad 0 \le s \le 1.$$

Domain decomposition	
N	accuracy
41	$9.132 \cdot 10^{-2}$
81	$7.649 \cdot 10^{-2}$

This is an equation with a weakly singular kernel. The Riemann scheme is not applicable to such problems and the kernel decomposition approach fails, for the diameter of the computed Fourier coefficients becomes too large.

Green's function kernel

$$x(s) = s^4 - 2s^3 + s + \pi \int_0^1 \left\{ \frac{1}{2}(s+t - |s-t|) - st \right\} x(t)\, dt, \quad 0 \le s \le 1.$$

Riemann approach		Kernel decomposition	
n	accuracy	n	accuracy
40	$1.8596 \cdot 10^{-1}$	9	$8.713 \cdot 10^{-3}$
80	$8.597 \cdot 10^{-2}$	27	$3.343 \cdot 10^{-3}$

Kernels with discontinuities in higher derivatives are typical of Green's function kernels.

Love's equation

$$x(s) = s^2 + \lambda \int_0^1 \frac{c}{c^2 + (s-t)^2} x(t)\, dt.$$

This problem, known as Love's equation (cf. Baker [3]), has been treated for different values of c and λ.

Riemann approach			
c	λ	n	Accuracy
0.1	-1.0	40	$5.270 \cdot 10^{-1}$
0.1	-1.0	80	$1.381 \cdot 10^{-1}$
1	-10	40	$1.235 \cdot 10^{-1}$
1	-10	80	$5.478 \cdot 10^{-2}$

For small values of c this equation becomes nearly singular, so that the decomposition scheme requires an unrealistic computational effort; thus, we display only the values for the Riemann algorithm.

Discontinuous kernel

$$x(s) = e^s + \int_0^1 k(s,t)\, x(t)\, dt, \quad 0 \le s \le 1$$

$$k(s,t) = \begin{cases} e^{s-t}, & 0 \le t \le s \\ 0, & \text{else}. \end{cases}$$

This is a Fredholm approach for a Volterra equation and, therefore, a kernel with a discontinuity along the line $s = t$.

Riemann approach		Kernel decomposition	
n	accuracy	n	accuracy
40	$1.825 \cdot 10^{-1}$	40	$1.100 \cdot 10^{-1}$
80	$9.177 \cdot 10^{-2}$	80	$5.241 \cdot 10^{-2}$

As in the above example, the method of kernel decomposition is uneffective.

Obviously, the most accurate procedure is the kernel decomposition, its results being superior to all the other numerical methods. But, in contrast to the Riemann scheme, a series expansion of the kernel or a sufficiently smooth kernel function (in order to determine a Taylor expansion with the help of automatic

differentiation) is necessary. The (validating) Riemann scheme is comparable with classical methods concerning accuracy, whereas, of course, the computation is as twice as much.

Method	Main area of application	Accuracy
Riemann approach	Continuous kernels	Low-mid high
Decomposition of the kernel	Kernels with series expansion	Mid-high high
Decomposition of the domain	Weakly continuous kernels	Low

Low, mid-high and high mean that the number of correct digits in the verified solution is about 1–2, 3–5 and from 6 upward.

REFERENCES

1. E. Adams and U. Kulisch, (Eds), *Scientific Computing with Automatic Result Verification*, Academic Press, 1993.
2. K. E. Atkinson, *A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind*, SIAM, Philadelphia, 1976.
3. C. Baker, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford, 1977.
4. H.-J. Dobner, *Verification Methods for Fredholm Integral Equations*, International Journal of Computer Mathematics **48** (1993), 251-261.
5. H.-J. Dobner, *Verified Solution of the Integral Equations for Two-Dimensional Dirichlet and Neumann Problems*, Computing Suppl. **9** (1933); 33–43, (1993) 251-261.
6. S. Fenyö and H. W. Stolle, *Theorie und Praxis der Linearen Integralgleichungen* 2, Birkhäuser, Basel–Boston–Stuttgart, 1984.
7. R. Hammer, M. Hocks, U. Kulisch and D. Ratz, *Numerical Toolbox for Verified Computing I*, Springer Series in Computational Mathematics, Vol. 21, Springer Verlag, Berlin–Heidelberg, 1993.
8. W. Klein, *Enclosure Methods for Linear and Nonlinear Systems of Fredholm Integral Equations of the Second Kind*, Scientific Computing with Automatic Result Verification, E. Adams, U. Kulisch (Eds.), Academic Press, 1993, pp. 255–282.
9. S. G. Michlin, *Vorlesungen über lineare Integralgleichungen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1962.
10. R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1966.
11. A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, 1990.
12. H. Ratschek, *Die Subdistrizbutivität der Intervallarithmetik*, ZAMM **51** (1971), 189-192.
13. U. Storck, *Numerical Integration in Two Dimensions with Automatic Result Verification*, In: Scientific Computing with Automatic Result Verification, E. Adams, U. Kulisch (Eds.), Academic Press, 1993, pp. 88–224.

University of Karlsruhe
76128 Karlsruhe Germany