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TESTS OF EFFICIENCY FOR A DISCRETE MULTICRITERIA OPTIMIZATION PROBLEM*

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For the multicriteria optimization problem, various necessary and sufficient conditions for a solution to be efficient are well-known (results by T. C. Koopmans, A. Wald, S. Karlin, L. Hurwicz, A. M. Geoffrion, Yu. B. Germeyir, P. L. Yu, V. V. Podinovskey, V. D. Nogin, A. Charnes, W. W. Cooper, R. E. Burkard, and others, see, for example, [1–13]). These conditions are the bases of the elaboration of numerical algorithms for finding efficient solutions.

In this paper we give some new simultaneously necessary and sufficient conditions for a vector valuation to be efficient in a multicriteria optimization problem with a finite criterion-valued set.

Let the multiobjective function

$$y=(y_1, y_2, \ldots, y_n): X \to \mathbb{R}^n, \ n \geq 2,$$

with the particular criteria

$$y_i(x) \to \min_X \ \forall i \in N_n = \{1,2,\ldots,n\},$$

be defined on the arbitrary set X of admissible solutions.

Further, we assume that the criterion-valued set

$$Y = y(X) = \{y \in \mathbb{R}^n : y = y(x), \ x \in X\}$$

is finite.

We consider the n-criterion discrete problem of search of the Pareto set

$$P(Y) = \{y \in Y : \pi(y) = \emptyset\},$$

where

$$\pi(y) = \{y' \in Y : y \geq y', \ y \neq y'\}.$$

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The elements of this set are called the efficient valuations or the Pareto optimal valuations.

The traditional method for finding efficient valuations in the n-criterion problem is the linear convolution of criteria ([7] and [8]), which can be expressed in the form of the following almost evident inclusion

(1) $$\Lambda(Y) \subseteq P(Y),$$

where

$$\Lambda(Y) = \bigcup_{\lambda \in \Lambda_n} \Lambda(Y, \lambda),$$

$$\Lambda_n = \{\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i > 0 \; \forall i \in N_n\},$$

$$\Lambda(Y, \lambda) = \operatorname{argmin}\{\langle \lambda, y\rangle : y \in Y\},$$

$\langle \lambda, y\rangle = \sum_{i=1}^{n} \lambda_i y_i(x)$ is a linear convolution of the particular criteria and argmin $\{\cdot\}$ is the set of all optimal solutions of the corresponding minimization problem.

In other words, for any vector $\lambda \in \Lambda_n$ the optimal valuation of the one-criterion problem

(2) $$\min\{\langle \lambda, y\rangle : y \in Y\}$$

is an efficient valuation of the n-criterion problem.

Unfortunately, there exist efficient valuations of some n-criterion problems that are not optimal valuations of problem (2) for any vector $\lambda \in \Lambda_n$, i.e., these efficient valuations cannot be found by means of a linear convolution of particular criteria (see, for example, [7], [11] and 14–16]).

We shall introduce another set of elements

$$C(Y) = \{y \in Y : \xi(y) = \emptyset\},$$

where

$$\xi(y) = \{y' \in \operatorname{conv} Y : y \geq y', \; y \neq y'\}$$

and conv Y is the convex hull of the criterion-valued set Y in $\mathbb{R}^n$.

LEMMA 1. [10, 11]

$$C(Y) = \Lambda(Y).$$

LEMMA 2. *Let* $Y \subset \mathbb{R}^n_+$. *If the formula*

(3) $$\forall i \in N_n \;\; \forall y, y' \in X \; (y_i < y'_i \Rightarrow n y_i \leq y'_i)$$

is true, then $P(Y) = \Lambda\,(Y)$.

Proof. According to (1) to prove the Lemma it is enough to show the inclusion $P(Y) \subseteq \Lambda(Y)$. We shall prove it by contradiction. If $y^o \in P(Y) \setminus \Lambda(Y)$, then, by Lemma 1, the vector $y^o \notin C(Y)$, i.e., $\xi(y^o) \neq \emptyset$. So there is a vector $y^* \in \xi(y^o)$ complying with the condition

$$\exists y \in \operatorname{conv} Y : y^* \geq y, \; y^* \neq y,$$

as the polyhedrom conv $Y \subset \mathbb{R}^n_+$ is bounded and extended. This means that the vector y^* belongs to several bound H, $\dim H \leq n-1$, of the polyhedron conv Y. Whence on the bases of the known relations

$$H = \operatorname{conv} \operatorname{vert} H, \quad \operatorname{vert} H \subseteq Y$$

(see [17, Theorem 2.2 and Corollary 2.4 in Chapter 1]) and Caratheodory theorem (see [17, Theorem 1.8 in Chapter 1]) the vector $y^* = (y_1^*, y_2^*, \ldots, y_n^*)$ is representable in the form

(4) $$y^* = \sum_{j=1}^{k} \lambda_j y^j, \quad k \leq n,$$

where $y^j \in Y, y^j \neq y^o \; \forall j \in N_k$ (as $y^* \in \xi(y^o)$), $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda_k$.

Therefore, since $y^o \in P(Y)$, for any number $j \in N_k$ there is an index $i(j) \in N_n$ complying with the inequality

$$y^o_{i(j)} < y^j_{i(j)}.$$

Hence, on the account of the implication of formula (3), the inequalities

(5) $$n y^o_{i(j)} \leq y^j_{i(j)} \quad \forall j \in N_k$$

are valid.

Further, taking into account relation (4) and inclusions $Y \subset \mathbb{R}^n_+$ and $y^* \in \xi(y^o)$, we derive

$$y^o_{i(j)} \geq y^*_{i(j)} = \sum_{s=1}^{k} \lambda_s y^s_{i(j)} \geq \lambda_j y^j_{i(j)} \quad \forall j \in N_k;$$

moreover, there is such an index $j^* \in N_k$ for which the strict inequality

$$y^o_{i(j^*)} > \lambda_{j^*} y^{j^*}_{i(j^*)}$$

is valid.

Hence, by (5) we obtain

$$\lambda_j \leq \frac{1}{n} \forall j \in N_k, \lambda_{j^*} < \frac{1}{n}.$$

Then we have

$$\sum_{j=1}^{k} \lambda_j < \frac{k}{n} \leq 1,$$

but $\sum_{j=1}^{k} \lambda_j = 1$, because $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda_k$, a contradiction which proves Lemma 2.

Let

(6) $$\alpha = n^{1/\Delta},$$

where $\Delta = \min\{y_i - y_i' > 0 : y, y' \in Y, i \in N_n\}$.

THEOREM 1. *Let* $|Y| < \infty$. *For any number* $a \geq \alpha$ *the vector* $\tilde{y} \in Y$ *is an efficient valuation iff there is a vector* $\lambda \in \Lambda_n$, *such that*

$$\sum_{i=1}^{n} \lambda_i a^{\tilde{y}_i} = \min\left\{\sum_{i=1}^{n} \lambda_i a^{y_i} : y \in Y\right\}.$$

Proof. From formula (6) we derive the proposition

$$\forall a \geq \alpha \forall i \in N_n \forall y, y' \in Y \ (a^{y_i} < a^{y_i'} \Rightarrow n a^{y_i} \leq a^{y_i'}).$$

So, on the basis of Lemma 2, for any $a \geq \alpha$

$$P(Y_a) = \Lambda(Y_a).$$

Here

$$Y_a = \{y \in \mathbb{R}^n : y = g_a(x), \ x \in X\},$$

$$g_a(x) = (a^{f_1(x)}, a^{f_2(x)}, \ldots, a^{f_n(x)}).$$

This is equivalent to the conclusion of Theorem 1.

Let

$$\beta = \log n / \log \min\left\{\frac{y_i'}{y_i} > 1 : y, y' \in Y, i \in N_n\right\}.$$

THEOREM 2. *Let* $|Y| < \infty$ *and* $Y \subset \mathbb{R}^n_+$. *For any number* $b \geq \beta$ *the vector* $\tilde{y} \in Y$ *is an efficient valuation iff there is a vector* $\lambda \in \Lambda_n$, *such that*

$$\sum_{i=1}^{n} \lambda_i \tilde{y}_i^b = \min\left\{\sum_{i=1}^{n} \lambda_i y_i^b : y \in Y\right\}.$$

Proof. The proof is analogous to that of Theorem 1.

In conclusion, for comparison, we state the known result which is analogous to Theorem 2.

THEOREM 3. [12] (*see also* [16]). *Let* $|Y| < \infty$ *and* $Y \subset \mathbb{R}^n_+$. *There is a vector* $k = (k_1, k_2, \ldots, k_n), k_i > 0 \ \forall i \in N_n$, *such that the vector* $\tilde{y} \in Y$ *is an efficient valuation iff there is a vector* $\lambda \in \Lambda_n$, *such that*

$$\sum_{i=1}^{n} \lambda_i \tilde{y}_i^{k_i} = \min\left\{\sum_{i=1}^{n} \lambda_i y_i^{k_i} : y \in Y\right\}.$$

In [12] the components of the vector $k = (k_1, k_2, \ldots, k_n)$ have been constructed with the help of the following recursion procedure:

$$k_1 = 1, \quad k_i > \frac{\log \frac{a_i}{b_i}}{\log c_i}, \ i = 2, 3, \ldots, n,$$

where

$$a_i = \max\left\{\sum_{j=1}^{i-1} \lambda_j (y_j^{k_j} - \dot{y}_j^{k_j}) : y, \dot{y} \in Y; \ (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}) \in \Lambda_{i-1}\right\},$$

$$b_i = \min\left\{\sum_{j=1}^{i-1} \lambda_j (y_j^{k_j} - \dot{y}_j^{k_j}) > 0 : y, \dot{y} \in Y; \ (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}) \in \Lambda_{i-1}\right\},$$

$$c_i = \min\left\{\frac{y_i}{\dot{y}_i} > 1 : y, \dot{y} \in Y\right\},$$

$\lambda_1 = 1$ if $i = 2$.

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