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ON AN ITERATIVE METHOD WITH MORE STEPS USING AN ALGEBRAIC CONDITION

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Let (X,ρ) be a metric space and $B\subset X, B\neq\emptyset$ a sphere. We consider the equation $f(x)=y(*)$, where $f:B\to X$ is a given function and $y\in X$ a fixed element. We suppose that we can put in correspondence to the equation (*) the new equation $\varphi(x)=x(**)$, where $\varphi:B\to X$ is a function, such that the solution of the equation (*) is a solution for the equation (**) and conversely. We say that the solution of the equation (**) is a fixed point for φ.

In order to solve the equation (**) we suppose that there exists a function $F:B^n\to X$, where $n\geq 1$ is a natural number, such that the restriction of F to the diagonal of the set B^n coincides with φ, i.e., $F(x,x,\ldots,x)=\varphi(x)$ for every $x\in B(***)$. Then we take the following iterative method with n steps:

$$x_n=F(x_0,x_1,\ldots,x_{n-1}) \quad \text{and} \quad x_{k+n}=F(x_k,x_{k+1},\ldots,x_{k+n-1})$$

for every $k=1,2,\ldots$ and $x_0,x_1,\ldots,x_{n-1}\in B$.

For assuring the convergence of the obtained sequence $\{x_k\}_{k\in\mathbb{N}}$ to the fixed point of the function φ we have the following known result:

THEOREM 1. *If (X,ρ) is a complete metric space and the function F satisfies the following conditions:*

i) *transforms the set B^n into B,*

ii) *verifies the condition (***),*

iii) *for every $y_1,y_2,\ldots,y_n,y_{n+1}\in B$ the function F satisfies the inequality*

$$\rho(F(y_1,y_2,\ldots,y_n),\ F(y_2,y_3,\ldots,y_{n+1}))\leq$$

$$\leq m_1\cdot\rho(y_1,y_2)+m_2\cdot\rho(y_2,y_3)+\ldots+m_n\cdot\rho(y_n,y_{n+1}),$$

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where $m_1, m_2, \ldots, m_n \geq 0$ *are real numbers such that* $m_1 + m_2 + \ldots + m_n < 1$, *then the sequence* $\{x_k\}_{k \in \mathbb{N}}$ *obtained by the iterative method with n steps is convergent for every* $x_0, x_1, \ldots, x_{n-1} \in B$ *and if we denote* $x^* = \lim_{k \to \infty} x_k$, *then* x^* *is the unique fixed point for* φ *in B.*

This theorem appears for the real case, when $X = \mathbb{R}$ in [1]. It is an easy exercise to transpose the statement and the proof of the theorem from [1] for metric spaces. When F is defined on the whole space X, then condition i) from Theorem 1 is superfluous. In this case theorem 1 appears in [2] with another proof.

The purpose of this paper is to replace the theoretical condition i) from Theorem 1 by a sufficient algebraic condition that we can verify concretely.

THEOREM 2. *Let* (X, ρ) *be a complete space,* $x_0 \in X$ *a fixed element and* $B = B(r_0, r) = \{x \in X / \rho(x, x_0) \leq r\}$ *a given sphere, where* $r > 0$. *If:*

i) *the function* $F : B^n \to X$ *satisfies condition* (***),

ii) *for every* $y_1, y_2, \ldots, y_n, y_{n+1} \in B$ *the function F verifies the inequality*

$$\rho(F(y_1, y_2, \ldots, y_n),\ F(y_2, y_3, \ldots, y_{n+1})) \leq$$

$$\leq m_1 \cdot \rho(y_1, y_2) + m_2 \cdot \rho(y_2, y_3) + \ldots + m_n \cdot \rho(y_n, y_{n+1}),$$

where $m_1, m_2, \ldots, m_n \geq 0$ *are real numbers such that* $m_1 + m_2 + \ldots + m_n < 1$,

iii) *the complex numbers* $z_1, z_2, \ldots, z_n$ *are the roots (with multiplicity one) of the equation* $P(t) = t^n - m_n t^{n-1} - \ldots - m_2 \cdot t - m_1 = 0$ *and* $\beta_1, \beta_2, \ldots, \beta_n$ *are the solution of the Vandermonde-type system of*

$$\begin{cases} \beta_1 + \beta_2 + \ldots + \beta_n = \rho(x_1, x_0) \\ \beta_1 z_1 + \beta_2 z_2 + \ldots + \beta_n z_n = \rho(x_2, x_1) \\ \vdots \\ \beta_1 z_1^{n-1} + \beta_2 z_2^{n-1} + \ldots + \beta_n z_n^{n-1} = \rho(x_n, x_{n-1}) \end{cases}$$

with $x_1, x_2, \ldots, x_{n-1}, x_n \in B$, *where* $x_n = F(x_0, x_1, \ldots, x_{n-1})$ *so that*

$$\frac{|\beta_1|}{1 - |z_1|} + \frac{|\beta_2|}{1 - |z_2|} + \ldots + \frac{|\beta_n|}{1 - |z_n|} \leq \frac{r}{2},$$

then the sequence $\{x_k\}_{k \in \mathbb{N}}$ *obtained by the iterative method with n steps is well defined, i.e., the terms of the sequence are in B, it is convergent and if we denote* $x^* = \lim_{k \to \infty} x_k$, *then* x^* *is the unique fixed point for* φ *in B.*

Proof. We show, using the mathematical induction method, that all terms of the sequence $\{x_k\}_{k \in \mathbb{N}}$ are in B. The demonstration is identical for every $k \in \mathbb{N}$, when we want to show that $x_{k+n+1} \in B$ if $x_k, \ldots, x_{k+n} \in B$. From ii) we obtain

$$\rho(x_{k+n+1}, x_{k+n}) = \rho(F(x_{k+1}, x_{k+2}, \ldots, x_{k+n}),\ F(x_k, x_{k+1}, \ldots, x_{k+n-1})) \leq$$

$$\leq m_1 \cdot \rho(x_{k+1}, x_k) + m_2 \cdot \rho(x_{k+2}, x_{k+1}) + \ldots + m_n \cdot \rho(x_{k+n}, x_{k+n-1}).$$

For every $k \in \mathbb{N}$ we denote $a_k = \rho(x_{k+1}, x_k) \geq 0$. So we obtain the inequalities

$$a_{k+n} \leq m_1 a_k + m_2 a_{k+1} + \ldots + m_n a_{k+n-1}.$$

We generate the new sequence $\{a'_k\}_{k \in \mathbb{N}}$ using the equalities

$$a'_{k+n} = m_1 a'_k + m_2 a'_{k+1} + \ldots + m_n a'_{k+n-1},$$

where $a'_0 = a_0, a'_1 = a_1, \ldots, a'_{n-1} = a_{n-1}$. It is easy to verify by mathematical induction that $a_k \leq a'_k$ for every $k \in \mathbb{N}$. Now we determine the general term a'_k using the linear recurrence

$$a'_{k+n} - m_1 a'_k + m_2 a'_{k+1} - \ldots - m_n a'_{k+n-1} = 0.$$

We consider the corresponding characteristic equation

$$t^n - m_n t^{n-1} - \ldots - m_2 t - m_1 = 0.$$

All the roots of this equation have modulus less than one. Indeed, if $\alpha \in \mathbb{C}$ is a root and we suppose that $|\alpha| \geq 1$, then

$$\alpha^n - m_n \alpha^{n-1} - \ldots - m_2 \alpha - m_1 = 0, \quad \text{so}$$

$$\alpha^n = m_n \alpha^{n-1} + \ldots + m_2 \alpha + m_1.$$

Dividing the equation by α^n, we obtain

$$1 = \frac{m_n}{\alpha} + \ldots + \frac{m_2}{\alpha^{n-1}} + \frac{m_1}{\alpha^n}, \quad \text{so}$$

$$1 = \left| \frac{m_n}{\alpha} + \ldots + \frac{m_2}{\alpha^{n-1}} + \frac{m_1}{\alpha^n} \right| \leq \frac{m_n}{|\alpha|} + \ldots + \frac{m_2}{|\alpha|^{n-1}} + \frac{m_1}{|\alpha|^n} \leq$$

$$\leq m_1 + m_2 + \ldots + m_n < 1,$$

which is a contradiction.

Without losing the generality of our problem, we can suppose that the roots of the characteristic equation are different pairwise.

Indeed, if the root t^* has the multiplicity at least two, then it verifies the equations

$$t^{*n} - m_n t^{*n-1} - \ldots - m_2 t^* - m_1 = 0 \quad \text{and}$$

$$n \cdot t^{*n-1} - (n-1) \cdot m_n \cdot t^{*n-2} - \ldots - m_2 = 0.$$

If $t_1, t_2, \ldots, t_{n-1} \in \mathbb{C}$ are the roots of the derived equation, it is sufficient to change the value m_1 in order to make sure that they will not be roots of the characteristic equation. Consequently, we choose $\alpha_n = m_n, \ldots, \alpha_2 = m_2$ and $\alpha_1 \geq 0$ such that the inequality $\alpha_1 + \alpha_2 + \ldots + \alpha_n < 1$ remains true and $\alpha_1 > m_1$ and $\alpha_1 \notin \{t_i^n - m_n t_i^{n-1} - \ldots - m_2 \cdot t_i \mid i = \overline{1, n-1}\}$. Such a choice of α_1 is possible and does not modify the essence of our problem.

So we consider the recurrence

$$a''_{k+n} = \alpha_1 a''_k + \alpha_2 a''_{k+1} + \ldots + \alpha_n a''_{k+n-1},$$

where $a''_0 = a'_0, a''_1 = a'_1, \ldots, a''_{n-1} = a'_{n-1}$. By mathematical induction it is easy to verify that $a'_k \leq a''_k$ for every $k \in \mathbb{N}$ and from $a_k \leq a'_k$ we obtain that $a_k < a''_k$ for every $k \in \mathbb{N}$. Without losing the generality of our problem, if $z_1, z_2, \ldots, z_n$ are the pairwise different roots of the new characteristic equation

$$z^n - \alpha_n z^{n-1} - \ldots - \alpha_2 z - \alpha_1 = 0,$$

then we can determine the values a''_k in the following form

$$a''_k = \beta_1 z_1^k + \beta_2 z_2^k + \ldots + \beta_n z_n^k,$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are the solution of the following linear Vandermonde-type system:

$$\begin{cases} \beta_1 + \beta_2 + \ldots + \beta_n = a''_0 = a_0 = \rho(x_1, x_0) \\ \beta_1 z_1 + \beta_2 z_2 + \ldots + \beta_n z_n = a''_1 = a_1 = \rho(x_2, x_1) \\ \vdots \\ \beta_1 z_1^{n-1} + \beta_2 z_2^{n-1} + \ldots + \beta_n z_n^{n-1} = a''_{n-1} = a_{n-1} = \rho(x_n, x_{n-1}), \end{cases}$$

where $x_n = F(x_0, x_1, \ldots, x_{n-1})$.

Because the Vandermonde determinant of the linear system is different from zero, there exist the values $\beta_i, i = \overline{1, n}$. Consequently, $a_k \leq \beta_1 z_1^k + \beta_2 z_2^k + \ldots + \beta_n z_n^k$ for every $k \in \mathbb{N}$. Now we show that $x_{k+n+1} \in B$:

$$\rho(x_{k+n+1}, x_0) \leq \rho(x_{k+n+1}, x_{k+n}) + \ldots + \rho(x_1, x_0) \leq$$

$$\leq \beta_1 (z_1^{n+k} + \ldots + z_1 + 1) + \beta_2 (z_2^{n+k} + \ldots + z_2 + 1) + \ldots + \beta_n (z_n^{n+k} + \ldots + z_n + 1) =$$

$$= \beta_1 \cdot \frac{1 - z_1^{n+k+1}}{1 - z_1} + \beta_2 \cdot \frac{1 - z_2^{n+k+1}}{1 - z_2} + \ldots + \beta_n \cdot \frac{1 - z_n^{n+k+1}}{1 - z_n} =$$

The previous expression is a real number, so

$$= \left| \beta_1 \cdot \frac{1 - z_1^{n+k+1}}{1 - z_1} + \beta_2 \cdot \frac{1 - z_2^{n+k+1}}{1 - z_2} + \ldots + \beta_n \cdot \frac{1 - z_n^{n+k+1}}{1 - z_n} \right| =$$

$$\leq |\beta_1| \cdot \frac{|1 - z_1^{n+k+1}|}{|1 - z_1|} + |\beta_2| \cdot \frac{|1 - z_2^{n+k+1}|}{|1 - z_2|} + \ldots + |\beta_n| \cdot \frac{|1 - z_n^{n+k+1}|}{|1 - z_n|} \leq$$

$$\leq |\beta_1| \cdot \frac{2}{1 - |z_1|} + |\beta_2| \cdot \frac{2}{1 - |z_2|} + \ldots + |\beta_n| \cdot \frac{2}{1 - |z_n|} \leq r.$$

Thus, it is possible to define the sequence $\{x_k\}_{k \in \mathbb{N}}$ by using the function F.

In the sequel, we show that $\{x_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence. Indeed:

$$\rho(x_{k+p}, x_k) \leq \rho(x_{k+p}, x_{k+p-1}) + \rho(x_{k+p-1}, x_{k+p-2}) + \ldots + \rho(x_{k+1}, x_k) \leq$$

$$\leq \beta_1 \cdot (z_1^{k+p-1} + \ldots + z_1^k) + \beta_2 \cdot (z_2^{k+p-1} + \ldots + z_2^k) + \ldots + \beta_n \cdot (z_n^{k+p-1} + \ldots + z_n^k) =$$

$$= \beta_1 z_1^k \cdot \frac{1 - z_1^p}{1 - z_1} + \beta_2 z_2^k \cdot \frac{1 - z_2^p}{1 - z_2} + \ldots + \beta_n z_n^k \cdot \frac{1 - z_n^p}{1 - z_n}.$$

This last expression is less than every small $\varepsilon > 0$ if $k \in \mathbb{N}$ is sufficiently large, for every $p \in \mathbb{N}$. But X is a complete metric space, so there exists $\lim_{k \to \infty} x_k = x^* \in B$ because B is a closed sphere. After ii) F is a Lipschitzian function, so it is continuous in every argument. Taking the limit in the recurrence relation and using i), we obtain $x^* = F(x^*, \ldots, x^*) = \varphi(x^*)$. This fixed point is unique in B. Indeed, if $y^* \in B$ is another fixed point for φ, then $\rho(x^*, y^*) =$ $= \rho(\varphi(x^*), \varphi(y^*)) = \rho(F(x^*, \ldots, x^*), F(y^*, \ldots, y^*)) \leq m_1 \cdot \rho(x^*, y^*) + m_2 \rho(x^*, y^*) +$

$+\ldots+m_n\cdot\rho(x^*,y^*)<\rho(x^*,y^*)$, which means a contradiction because m_1+m_2+ $+\ldots+m_n<1$. Q.E.D.

For $n=1$ from Theorem 2 we obtain the following

THEOREM 3. *Let (X, ρ) be a complete metric space,* $x_0\in X$ *a fixed element, and* $B=B(x_0,r)=\{x\in X\mid\rho(x,x_0)\le r\}$ *a given sphere, where* $r>0$. *If the function* $\varphi:B\to X$ *verifies the following conditions*

i) $\rho(\varphi(y_1),\varphi(y_2))\le m\cdot\rho(y_1,y_2)$ *for every* $y_1,y_2\in B$ *and* $0\le m<1$,

ii) $\rho(\varphi(x_0),x_0)=\rho(x_1,x_0)\le(1-m)\cdot\frac{r}{2}$,

then the sequence $\{x_k\}_{k\in\mathbb{N}}$ *obtained by the iterative method* $x_{k+1}=\varphi(x_k)$ *is well defined, i.e., the terms of the sequence are in B, it is convergent and if we denote* $x^*=\lim_{k\to\infty}x_k$, *then* x^* *is the unique fixed point for φ in B.*

Proof. The condition ii) assures that $x_1\in B$ because

$$\rho(x_1,x_0)\le(1-m)\cdot\frac{r}{2}\le r.$$

We apply Theorem 2. In this case we have $z-m=0, z_1=m, \beta_1=\rho(x_1,x_0)$, so the condition $\frac{|\beta_1|}{1-|z_1|}\le\frac{r}{2}$ gives us that $\rho(x_1,x_0)\le(1-m)\cdot\frac{r}{2}$.

Remark 1. Theorem 3 is true if instead of ii) we consider the condition $\rho(x_1,x_0)\le(1-m)\cdot r$ (see [1]).

Example 1. Further on we consider a numerical example.

Let $X=\mathbb{R}, x_0=0, r=1, B=B(0,1)=[-1,1], f:[-1,1]\to\mathbb{R}, f(x)=-\frac{1}{5}e^{-x}-x$.

Because $f(-1)>0, f(1)<0$ and f is continuous there exists root of the equation $f(x)=0$ in B. To solve this equation we take the function $\varphi:[-1,1]\to\mathbb{R}$, $\varphi(x)=\frac{1}{5}e^{-x}$ and we apply Theorem 3. Indeed, from Lagrange theorem we get the value $m:|\varphi(y_1)-\varphi(y_2)|=\left|\frac{1}{5}e^{y_1}-\frac{1}{5}e^{y_2}\right|=\frac{1}{5}e^{-q}\cdot|y_1-y_2|\le\frac{e}{5}\cdot|y_1-y_2|$, where $q\in[-1,1]$, so $m=\frac{e}{5}$ and $|\varphi(x_0)-x_0|=|x_1-x_0|=\frac{1}{5}\le\left(1-\frac{e}{5}\right)\cdot\frac{1}{2}=(1-m)\cdot\frac{r}{2}$.

For $n=2$ from Theorem 2 we obtain the following

THEOREM 4. *Let (X, ρ) be a complete metric space,* $x_0\in X$ *a fixed element and* $B=B(x_0,r)=\{x\in X\mid\rho(x,x_0)\le r\}$ *a given sphere, where* $r>0$. *If the function* $F:B^2\to X$ *verifies the following conditions:*

i) $F(x,x)=\varphi(x)$ *for every* $x\in B$,

ii) *there exist the constants* $m_1,m_2\ge 0, m_1+m_2<1$ *so that*

$$\rho(F(y_1,y_2),F(y_2,y_3))\le m_1\cdot\rho(y_1,y_2)+m_2\cdot\rho(y_2,y_3)$$

for every $y_1,y_2,y_3\in B$,

iii) *the real numbers* z_1, z_2 *are the roots of the equation* $z^2-m_2z-m_1=0$ *with* $z_1<z_2$ *so that* $|a_0z_2-a_1|\cdot(1-z_2)+|a_0z_1-a_1|\cdot(1-z_1)\le(z_2-z_1)(1-m_1-m_2)\cdot\frac{r}{2}$, *where* $x_1\in B(x_0,r)$, $x_2=F(x_0,x_1)\in B$, $a_0=\rho(x_0,x_1)$, $a_1=\rho(x_1,x_2)$, *then the sequence* $\{x_k\}_{k\in\mathbb{N}}$ *obtained by the iterative method with two steps is well defined, i.e., the terms of the sequence are in B, it is convergent and* $\lim_{k\to\infty}x_k=x^*$, *where* x^* *is the unique fixed point for φ in* $B:F(x^*,x^*)=\varphi(x^*)=x^*$.

Proof. In this case the characteristic equation has the form

$$z^2-m_2z-m_1=0,$$

with real roots: $z_1=\frac{m_2-\sqrt{m_2^2+4m_1}}{2}<z_2=\frac{m_2+\sqrt{m_2^2+4m_1}}{2}$. An elementary calculus gives us that $z_1\in[-1,0]$ and $z_2\in(0,1)$ if $m_1+m_2<1$. We determine the values β_1 and β_2 from the system:

$$\begin{cases}\beta_1+\beta_2=a_0=|x_1-x_0|\\ \beta_1z_1+\beta_2z_2=a_1=|x_2-x_1|.\end{cases}$$

The searched algebraic condition has the following form

$$\left|\frac{a_0z_2-a_1}{z_2-z_1}\right|\cdot\frac{2}{|1-z_1|}+\left|\frac{a_1-a_0z_1}{z_2-z_1}\right|\cdot\frac{2}{|1-z_2|}\le r, \text{ i.e.,}$$

$$|a_0z_2-a_1|\cdot(1-z_2)+|a_1-a_0z_1|\cdot(1-z_1)\le(z_2-z_1)(1-z_1)(1-z_2)\cdot\frac{r}{2}=$$

$$=(z_2-z_1)(1-m_1-m_2)\cdot\frac{r}{2}.\quad \text{Q.E.D.}$$

Example 2. Further on we consider an another numerical example. Let $X=\mathbb{R}$, $x_0=0$, $r=1$, $B=B(0,1)=[-1,1]$, $f:[-1,1]\to\mathbb{R}$, $f(x)=\frac{1}{10}e^{-x}-x$, $\varphi:[-1,1]\to\mathbb{R}$, $\varphi(x)=\frac{1}{10}e^{-x}$. Let us consider $F:B^2\to\mathbb{R}, F(x,y)=\frac{1}{20}e^{-x}+\frac{1}{20}e^{-y}$, so $F(x,x)=\varphi(x)$. From the Lagrange's mean value theorem we obtain the values m_1 and m_2:

$$|F(y_1,y_2)-F(y_2,y_3)|=$$

$$=\left|\left(\frac{1}{20}e^{-y_1}-\frac{1}{20}e^{-y_2}\right)+\left(\frac{1}{20}e^{-y_2}-\frac{1}{20}e^{-y_3}\right)\right|\le$$

$$\le\frac{1}{20}e^{q_1}\cdot|y_1-y_2|+\frac{1}{20}e^{q_2}\,|y_2-y_3|\le\frac{e}{20}|y_1-y_2|+\frac{e}{20}|y_2-y_3|,$$

where $q_1,q_2\in[-1,1]$, so $m_1=m_2=\frac{e}{20}$. The equation $z^2-m_2z-m_1=0$ has the form $20z^2-ez-e=0$ with roots $z_1=\frac{e-\sqrt{e^2+80e}}{40}<z_2=\frac{e+\sqrt{e^2+80e}}{40}$. We choose $x_1=0\in[-1,1]$ and we calculate $x_2=F(x_0,x_1)=\frac{1}{10}$, $a_0=|x_0-x_1|=0$, $a_1=|x_1-x_2|=\frac{1}{10}$. So we verify the inequality iii) from Theorem 4:

$$\frac{1}{20}\left(1-\frac{e+\sqrt{e^2+80e}}{40}\right)+\frac{1}{20}\left(1-\frac{e-\sqrt{e^2+80e}}{40}\right)\le\frac{\sqrt{e^2+80e}}{20}\left(1-\frac{e}{10}\right)\cdot\frac{1}{2},\text{ i.e.,}$$

$$2(40-e)\le\sqrt{e^2+80e}\cdot(10-e),$$

which is true.

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