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DEGREE OF APPROXIMATION OF CONTINUOUS FUNCTIONS BY SOME SINGULAR INTEGRALS

SORIN G. GAL

1. INTRODUCTION

Let us denote

$$C_{2\pi} = \{f : \mathbb{R} \to \mathbb{R};\ f \text{ is } 2\pi\text{-periodic and continuous on } \mathbb{R}\}$$

and for $\alpha \in (0, 1]$

$$\text{Lip}\,\alpha = \{f \in C_{2\pi};\ \exists M > 0 \text{ with } |f(x) - f(y)| \le M|x-y|,\ \forall x, y \in \mathbb{R}\}.$$

For $f \in C_{2\pi}$ and $\xi > 0$ let us consider

$$P(x, \xi) = (2\xi)^{-1} \int_{-\infty}^{+\infty} f(x+t)\, e^{-|t|/\xi}\, \mathrm{d}t,$$

$$Q(x, \xi) = (\xi / \pi) \int_{-\pi}^{\pi} [f(x+t) / (t^2 + \xi^2)]\, \mathrm{d}t,$$

$$W(x, \xi) = (\pi\xi)^{-1/2} \int_{-\pi}^{\pi} f(x+t)\, e^{-t^2/\xi}\, \mathrm{d}t,$$

called the Picard, Poisson-Cauchy and Gauss-Weierstrass singular integrals, respectively (see, e.g., [8]).

For $f \in C_{2\pi}$ and $p \in \mathbb{N}$, the p-th modulus of smoothness of f is defined by (see, e.g., [5, p. 47])

$$\omega_p(f; t) = \sup\{|\Delta_h^p f(x)|;\ x, x+ph \in \mathbb{R},\ 0 \le h \le t\},$$

where

$$\Delta_h^p f(x) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} f(x+kh).$$

The modulus $\omega_1(f; t)$ is denoted by $\omega(f; t)$.

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Regarding the approximation by the previous singular integrals, the following estimates are obtained in ([8], [4]):

THEOREM 1.1. *If* $f \in C_{2\pi}$ *then as* $\xi \to 0+$ *we have*

$$\|f(x)-P(x,\xi)\| = \mathcal{O}(\omega(f;\xi)),\ \|f(x)-Q(x,\xi)\| = \mathcal{O}(\omega(f;\xi)\,|\ln(1/\xi)|),$$

$$\|f(x)-W(x,\xi)\| = \mathcal{O}(\omega(f;\xi)\,\xi^{-1/2}),$$

where the uniform $\|\cdot\|$ *is applied to x.*

The main purpose of this paper is to obtain error bounds in terms of higher order moduli of smoothness, $\omega_n(f;\xi)$, for approximation by singular integrals of the previous type. Thus, if $f^{(p)} \in \mathrm{Lip}\,\alpha$, then better approximation orders can be obtained. Also, in comparison with [8] and [4], the most estimates are obtained with explicit constants.

2. APPROXIMATION BY SINGULAR INTEGRALS OF PICARD-TYPE

Firstly, we shall improve the estimate in Theorem 1.1.

THEOREM 2.1. *If* $f \in C_{2\pi}$ *then we have*

$$\|f(x)-P(x,\xi)\| \le (5/2)\,\omega_2(f;\xi),\quad \forall \xi>0.$$

(ii) *If there exists* $f' \in \mathrm{Lip}\,\alpha$ *then*

$$\|f(x)-P(x,\xi)\| \le (5/2)\,\xi^{1+\alpha},\quad \forall \xi>0.$$

Proof. (i) By the proof of Theorem 1 in [8] we have

$$P(x,\xi)-f(x) = (2\xi)^{-1}\int_0^{+\infty} \phi_x(t)\,e^{-t/\xi}\,dt,$$

where

$$\phi_x(t) = f(x+t)-2f(x)+f(x-t).$$

Hence

$$|P(x,\xi)-f(x)| \le (2\xi)^{-1}\int_0^{+\infty} |\phi_x(t)|\,e^{-t/\xi}\,dt \le (2\xi)^{-1}\int_0^{+\infty} \omega_2(f;t)\,e^{-t/\xi}\,dt =$$

$$= (2\xi)^{-1}\int_0^{+\infty} \omega_2(f;(t/\xi)\xi)\,e^{-t/\xi}\,dt \le (2\xi)^{-1}\int_0^{+\infty} [1+t/\xi]^2\,e^{-t/\xi}\,dt =$$

$$= (2\xi)^{-1}\,\omega_2(f;\xi)\,\xi\int_0^{+\infty} (1+2u+u^2)\,e^{-u}\,du = C\omega_2(f;\xi),$$

where by a simple calculus we have

$$C = \int_0^{+\infty} (1+2u+u^2)\,e^{-u}\,du/2 = 5/2.$$

Passing to supremum with $x \in \mathbb{R}$, we get the desired estimate.

(ii) If $f' \in \mathrm{Lip}\,\alpha$ then we get

$$\|f(x)-P(x,\xi)\| \le (5/2)\,\omega_2(f;\xi) \le (5/2)\,\xi\omega(f';\xi) \le (5/2)\,\xi^{1+\alpha}.$$

Remark. Obviously, the order of approximation in Theorem 2.1 (ii) cannot be obtained by Theorem 1.1.

Now, following the ideas in [5, p. 57, relation (7)], we shall generalize the Picard's singular integral in the following way.

For $p \in \mathbb{N}$ let us consider

$$P_p(x,\xi) = -(2\xi)^{-1}\sum_{k=1}^{p+1}(-1)^k\binom{p+1}{k}\int_{-\infty}^{+\infty} f(x+kt)\,e^{-|t|/\xi}\,dt,\quad \xi>0.$$

We shall prove

THEOREM 2.2. (i) *If* $f \in C_{2\pi}$ *then we have*

$$\|f(x)-P_p(x,\xi)\| \le \left[\sum_{k=0}^{p+1}\binom{p+1}{k}k!\right]\omega_{p+1}(f;\xi),\quad \forall \xi>0.$$

(ii) If there exist $f^{(p)} \in C_{2\pi}$ then we get

$$\|f(x)-P_p(x,\xi)\| \le \left[\sum_{k=0}^{p+1}\binom{p+1}{k}k!\right]\xi^p\omega(f^{(p)};\xi),\quad \forall \xi>0.$$

Proof. We have

$$f(x)-P_p(x,\xi) = f(x)\,(2\xi)^{-1}\int_{-\infty}^{+\infty} e^{-|t|/\xi}\,dt +$$

$$+(2\xi)^{-1}\int_{-\infty}^{+\infty}\left[\sum_{k=1}^{p+1}(-1)^k\binom{p+1}{k}\right] f(x+kt)\,e^{-|t|/\xi}\,dt =$$

$$= (2\xi)^{-1}\int_{-\infty}^{+\infty} (-1)^{p+1}\Delta_t^{p+1}f(x)\,e^{-|t|/\xi}\,dt,$$

wherefrom

$$|f(x)-P_p(x,\xi)|\le(2\xi)^{-1}\int_{-\infty}^{+\infty}\omega_{p+1}(f;|t|)\,e^{-|t|/\xi}dt=$$

$$=\xi^{-1}\int_0^{+\infty}\omega_{p+1}(f;t)\,e^{-|t|/\xi}dt=\xi^{-1}\int_0^{+\infty}\omega_{p+1}(f;(t/\xi)\xi)\,e^{-t/\xi}dt\le$$

(see [5, p. 48])

$$\le\xi^{-1}\omega_{p+1}(f;\xi)\int_0^{+\infty}(1+t/\xi)^{p+1}e^{-t/\xi}dt=$$

$$=\omega_{p+1}(f;\xi)\int_0^{+\infty}(1+u)^{p+1}e^{-u}\,du=C_{p+1}\omega_{p+1}(f;\xi),$$

where

$$C_{p+1}=\int_0^{+\infty}(1+u)^{p+1}e^{-u}du=\sum_{k=0}^{p+1}\binom{p+1}{k}k!.$$

Passing to supremum with $x\in\mathbb{R}$, we get the desired estimate.

By $\omega_{p+1}(f;\xi)\le\xi^p\omega(f^{(p)};\xi)$, (ii) is an immediate consequence of (i).

Remark. A natural question which arises refers to the construction of singular integrals of Picard type which approximate the continuous functions defined on compact intervals. Thus, for example, if f is continuous on [0, 1] (we write $f\in C[0,1]$), then we can define

$$L[f](x,\xi)=\xi^{-1}\int_0^{+\infty}f(xe^{-t})\,e^{-t/\xi}\,dt,\quad x\in[0,1],\ \xi>0.$$

In this case, the following pointwise estimate holds:

THEOREM 2.3. *If $f\in C[0,1]$ then*

$$|L[f](x,\xi)-f(x)|\le 4\omega(f;\xi x),\quad \forall x\in[0,1],\ \forall\xi>0.$$

where

$$\omega(f;t)=\sup\{|f(x)-f(y)|;\ |x-y|\le t,\ \ x,y\in[0,1]\}.$$

Proof. Denoting $e_i(t)=t^i,\ i=0,1,2,$ we get

$$L[e_0](x,\xi)=1,$$

$$L[e_1](x,\xi)=x\xi^{-1}\int_0^{+\infty}e^{-t(1+1/\xi)}\,dt=$$

$$=[x\xi^{-1}/(1+1/\xi)][-e^{-t(1+1/\xi)}\Big|_0^{+\infty}]=x/(\xi+1),$$

$$L[e_2](x,\xi)=x^2\xi^{-1}\int_0^{+\infty}e^{-t(2+1/\xi)}\,dt=x^2/(2\xi+1).$$

For fixed $x\in[0,1]$ we get

$$L[(e_1-x)^2](x,\xi)=x^2/(2\xi+1)-2x^2/(\xi+1)+x^2=$$

$$=2x^2\xi^2/[(\xi+1)(2\xi+1)]\le 2x^2\xi^2.$$

Now, taking into account that L is a positive linear operator, by [3, Theorem 2.3] we immediately obtain

$$|L[f](x,\xi)-f(x)|\le 2\omega(f;\sqrt{2}\cdot\xi x)\le 4\omega(f;\xi x),$$

which proves the theorem.

At the end of this section we shall extend the Picard's singular integral to functions of two variables, in the following way.

Let us consider

$$C_{2\pi,2\pi}=\{f:\mathbb{R}\times\mathbb{R}\to\mathbb{R};\ f \text{ is continuous on } \mathbb{R}\times\mathbb{R} \text{ and } 2\pi\text{-periodic in each variable}\},$$

$$\|f\|=\sup\{|f(x,y)|;\ x,y\in\mathbb{R}\},\quad \forall f\in C_{2\pi,2\pi},$$

$$\omega(f;\xi,\eta)=\sup\{|f(x+h,y+k)-f(x,y)|;\ 0\le h\le\xi,\ 0\le k\le\eta,\ x,y\in\mathbb{R}\},\quad \xi,\eta>0,$$

and for $f\in C_{2\pi,2\pi}$

$$P(x,y,\xi,\eta)=(4\xi\eta)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x+t,y+s)\,e^{-|t|/\xi}\cdot e^{-|s|/\eta}\,dt\,ds,\quad \xi,\eta>0.$$

We shall prove

THEOREM 2.4. *If $f\in C_{2\pi,2\pi}$ then we have*

$$\|f(x,y)-P(x,y,\xi,\eta)\|\le 3\omega(f;\xi,\eta),\quad \forall\ \xi,\eta>0,$$

where the uniform norm $\|\cdot\|$ is applied to x and y.

Proof. We have

$$|P(x,y,\xi,\eta)-f(x,y)|=$$

$$=(4\xi\eta)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[f(x+t,y+s)-f(x,y)]\,e^{-|t|/\xi}\cdot e^{-|s|/\eta}\,dt\,ds\le$$

$$=(4\xi\eta)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\omega(f;|t|,|s|)\,e^{-|t|/\xi}\cdot e^{-|s|/\eta}\,dt\,ds=$$

$$=(\xi\eta)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\omega(f;(t/\xi)\,\xi,(s/\eta)\,\eta)\,e^{-t/\xi}\cdot e^{-s/\eta}\,dt\,ds\le$$

$$\le(\xi\eta)^{-1}\omega(f;\xi,\eta)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[1+t/\xi+s/\eta]\,e^{-t/\xi}\cdot e^{-s/\eta}\,dt\,ds=3\omega(f;\xi,\eta),$$

wherefrom passing to supremum with $x,y\in\mathbb{R}$, we get our estimate.

3. POISSON-CAUCHY AND GAUSS-WEIERSTRASS-TYPE INTEGRALS

Some ideas in the previous section will be considered in the case of the Poisson-Cauchy and Gauss-Weierstrass singular integrals, too.

Firstly, we shall prove

THEOREM 3.1. (i) *If* $f\in C_{2\pi}$ *then we have*

(1) $$\|f(x)-Q(x,\xi)\|\le[1+(1/\pi)\ln(\pi^2+1)]\,\xi^{-1}\omega_2(f;\xi)+(2/\pi^2)\,\xi\|f\|,\quad \xi\in(0,1]$$

and

(2) $$\|f(x)-W(x,\xi)\|\le(1/\sqrt{\pi})\,(\sqrt{\pi}/2+1+\sqrt{\pi}/4)\,\xi^{-1}\omega_2(f;\xi)+(\xi/\pi^{5/2})\,\|f\|,\quad \xi\in(0,1]$$

If, moreover, $f\not\equiv C$ *(constant) then as* $\xi\to 0+$ *we get*

$$\|f(x)-Q(x,\xi)\|=\mathcal{O}(\xi^{-1}\omega_2(f;\xi))$$

$$\|f(x)-W(x,\xi)\|=\mathcal{O}(\xi^{-1}\omega_2(f;\xi)).$$

(ii) If $f\not\equiv C$ (constant) and $f'\in \mathrm{Lip}\,\alpha$ then as $\xi\to 0+$ we have

$$\|f(x)-Q(x,\xi)\|=\mathcal{O}(\xi^{\alpha})$$

and

$$\|f(x)-W(x,\xi)\|=\mathcal{O}(\xi^{\alpha}).$$

Proof. (i) By the proof of Theorem 1 in [8] (relations (4.3), (4.4), (4.8) and (4.9)) we get

$$Q(x,\xi)-f(x)=(\xi/\pi)\int_0^{\pi}[\phi_x(t)/(t^2+\xi^2)]\,dt-f(x)\,E(\xi),$$

$$W(x,\xi)-f(x)=(\pi\xi)^{-1/2}\int_0^{\pi}\phi_x(t)\,e^{-t^2/\xi}\,dt-R(x,\xi),$$

where for all $\xi>0$ we have

$$|E(\xi)|=E(\xi)=1-(2\xi/\pi)\int_0^{\pi}dt/(t^2+\xi^2)=1-(2/\pi)\,\mathrm{arctg}\,(\pi/\xi)\le(2/\pi^2)\,\xi$$

and

$$|R(x,\xi)|\le(\sqrt{\pi})^{-1}\,\|f\|\,e^{-\pi^2/\xi}\le(\sqrt{\pi})^{-1}\,(\xi/\pi^2)\,\|f\|.$$

Hence, for $x\in\mathbb{R}$ and $\xi>0$ we obtain

(3) $$|Q(x,\xi)-f(x)|=(\xi/\pi)\int_0^{\pi}[\omega_2(f;t)/(t^2+\xi^2)]\,dt+\|f\|\cdot|E(\xi)|$$

and

(4) $$|W(x,\xi)-f(x)|\le(\pi\xi)^{-1/2}\int_0^{\pi}\omega_2(f;t)\,e^{-t^2/\xi}\,dt+|R(x,\xi)|.$$

But

$$(\xi/\pi)\int_0^{\pi}[\omega_2(f;t)/(t^2+\xi^2)]\,dt=(\xi/\pi)\int_0^{\pi}[\omega_2(f;(t/\xi)\,\xi)/(t^2+\xi^2)]\,dt\le$$

$$\le(\xi/\pi)\,\omega_2(f;\xi)\int_0^{\pi}[(1+t/\xi)^2/(t^2+\xi^2)]\,dt=$$

$$=(\xi/\pi)\,\omega_2(f;\xi)\int_0^{\pi}\{1/\xi^2+2t/[\xi(t^2+\xi^2)]\}\,dt=$$

$$=(\xi/\pi)\,\omega_2(f;\xi)[\pi/\xi^2+(1/\xi)\ln((\pi^2+\xi^2)/\xi^2)]=\omega_2(f;\xi)/\xi+$$

$$+(1/\pi)\,\omega_2(f;\xi)\ln((\pi^2+\xi^2)/\xi^2)\le[1+(1/\pi)\ln(\pi^2+1)]\,\xi^{-1}\omega_2(f;\xi),$$

for all $\xi \in (0,1]$, since it is easy to prove that

$$\ln[(\pi^2+\xi^2)/\xi^2] \le (1/\xi)\ln(\pi^2+1), \quad \forall \xi \in (0,1].$$

Then by (3) we immediately get (1).

Analogously, in the case of $W(x,\xi)$ we have

$$(\pi\xi)^{-1/2}\int_0^{\pi}\omega_2(f;t)\,e^{-t^2/\xi}\,dt \le (\pi\xi)^{-1/2}\omega_2(f;\xi)\int_0^{\pi}(1+t/\xi)^2 e^{-t^2/\xi}dt \le$$

$$\le (\pi\xi)^{-1/2}\omega_2(f;\xi)\left\{(\pi\xi)^{1/2}/2+1+\xi^{-1/2}\int_0^{+\infty}u^2 e^{-u^2}du\right\} =$$

(by [7, p. 17, Problem 1.40, c)])

$$= (\pi\xi)^{-1/2}\omega_2(f;\xi)\{(\pi\xi)^{1/2}/2+1+\xi^{-1/2}(\sqrt{\pi}/4)\} \le$$

$$\le (1/\sqrt{\pi})(\sqrt{\pi}/2+1+\sqrt{\pi}/4)\xi^{-1}\omega_2(f;\xi), \quad \text{for all } \xi \in (0,1],$$

which, together with (4), immediately proves (2).

The condition $\not\equiv C$ (constant) implies $\omega_2(f;\pi) \ne 0$. Indeed, if $\omega_2(f;\pi)=0$, then by [5, p. 52, Problem 4] we easily get that f is linear on each interval, which combined with $f \in C_{2\pi}$, implies the contradiction $f \equiv C$ (constant) on $\mathbb{R}$.

Then by [2, p. 488, Property 7] we get

$$\xi = \xi^{-1}\xi^2 = \xi^{-1}\mathcal{O}(\omega_2(f;\xi)) = \mathcal{O}(\xi^{-1}\omega_2(f;\xi)),$$

which, together with (1) and (2), immediately gives

$$\|f(x)-Q(x,\xi)\| = \mathcal{O}(\xi^{-1}\omega_2(f;\xi)),$$

$$\|f(x)-W(x,\xi)\| = \mathcal{O}(\xi^{-1}\omega_2(f;\xi)).$$

(ii) By (i) we get

$$\|f(x)-Q(x,\xi)\| = \mathcal{O}(\xi^{-1}\omega_2(f;\xi)) = \mathcal{O}(\xi^{-1}\xi\omega(f';\xi)) = \mathcal{O}(\xi^{\alpha}).$$

The proof in the case of $W(x,\xi)$ is entirely analogous.

Remark. Obviously, the estimates in Theorem 3.1 cannot be obtained by Theorem 1.1. On the other hand, note that the same condition $f \not\equiv C$ (constant) is necessary for the validity of the estimates in Theorem 1.1, too, concerning the approximation by $Q(x,\xi)$ and $W(x,\xi)$.

The method in [5, p. 57, relation (7)] can be used in the Poisson-Cauchy and Gauss-Weierstrass integrals, too. As, for example, the Gauss-Weierstrass singular integrals can be generalized by

$$W_p(x,\xi) = -[1/(2C(\xi))]\sum_{k=1}^{p+1}(-1)^k\binom{p+1}{k}\int_{-\pi}^{\pi}f(x+kt)\,e^{-t^2/\xi^2}dt,$$

where $p \in \mathbb{N} \cup \{0\}, \xi>0, r>p/2+2$ and $C(\xi)=\int_0^{\pi}e^{-t^2/\xi^2}dt$, then an analogue with Theorem 2.2 (in Section 2) can be proved in this case, too.

Firstly, we need the following.

LEMMA 3.2. *We have*

$$\xi\int_0^{\pi}e^{-u^2}du \le C(\xi) \le \xi\sqrt{\pi}/2, \quad 0<\xi\le 1.$$

Proof. We can write (see, e.g., [7, p. 17, Problem 1.40, c)])

$$\int_0^{\pi}e^{-t^2/\xi}dt = \xi\int_0^{\pi/\xi}e^{-u^2}du \le \xi\int_0^{+\infty}e^{-u^2}du = \xi\sqrt{\pi}/2, \quad \forall \xi>0.$$

On the other hand, for $\xi \le 1$ we get

$$\xi\int_0^{\pi/\xi}e^{-u^2}du \ge \xi\int_0^{\pi}e^{-u^2}du,$$

since $e^{-u^2}>0$ and $\pi/\xi \ge \pi$, which proves the lemma.

Similar with Theorem 2.2

THEOREM 3.3. *We have*

$$\|f(x)-W_p(x,\xi)\| = \mathcal{O}(\omega_{p+1}(f;\xi)), \forall 0<\xi\le 1. \tag{5}$$

Proof. We get

$$f(x)-W_p(x,\xi) = [1/(2C(\xi))]\int_{-\pi}^{\pi}(-1)^{p+1}\Delta_t^{p+1}f(x)\,e^{-t^2/\xi^2}dt,$$

which implies

$$|f(x)-W_p(x,\xi)| \le [1/C(\xi)]\int_0^{\pi}\omega_{p+1}(f;t)\,e^{-t^2/\xi^2}dt \le$$

$$\le [1/C(\xi)]\omega_{p+1}(f;\xi)\int_0^{\pi}[1+t/\xi]^{p+1}e^{-t^2/\xi^2}dt =$$

$$=[\xi / C(\xi)]\,\omega_{p+1}(f;\xi)\int_0^{\pi/\xi}[1+u]^{p+1}\,e^{-u^2}\,du \leq$$

(Lemma 3.2)

$$\leq \left[\xi \Big/ \left(\xi \int_0^{\pi} e^{-u^2}\,du\right)\right] \omega_{p+1}(f;\xi) \int_0^{+\infty}[1+u]^{p+1}\,e^{-u^2}\,du =$$

$$= \left[1 \Big/ \int_0^{\pi} e^{-u^2}\,du\right]\left[\int_0^{+\infty}[1+u]^{p+1}\,e^{-u^2}\,du\right]\omega_{p+1}(f;\xi) =$$

$$= \mathcal{O}(\omega_{p+1}(f;\xi)),\quad 0<\xi\leq 1,$$

which proves the theorem.

4. FINAL REMARKS

Remark 4.1. Related with $Q(x,\xi)$, the Poisson-Cauchy singular integral in Introduction, it is the well-known Poisson integral defined by

$$I(x,\xi)=(\xi/\pi)\int_{-\infty}^{+\infty}[f(x+t)/(t^2+\xi^2)]\,dt,\quad \xi>0.$$

As concerns this integral, Th. Angheluţă proved in [1] the estimate

$$\|f(x)-I(x,\xi)\| = \mathcal{O}(\omega(f;\xi)\,|\ln(1/\xi)|),\quad \text{as}\quad \xi\to 0+,\quad f\in C_{2\pi}.$$

Comparing with Theorem 1.1, we note that although $Q(x,\xi)$ and $I(x,\xi)$ differ in their limits of integration, they give the same order of approximation.

Remark 4.2. It is not difficult to verify that, for example, $Q(x,\xi)$ and $W(x,\xi)$ are positive linear operators on $C_{2\pi}$, satisfying the conditions in the classical Korovkin's result.

However, it is easy to verify that the estimates which can be derived by, e.g., [3] are weaker than those given by our previous results.

Remark 4.3. As regards the Poisson singular integral $I(x,\xi)$ in Remark 4.1, a saturation theorem is proved in [6]. Then it would be of interest to obtain saturation theorems for $P(x,\xi)$, $P_p(x,\xi)$, $Q(x,\xi)$, $W(x,\xi)$ and $W_p(x,\xi)$, too.

Remark 4.4. With respect to the Poisson singular integral $I(x,\xi)$, it is known the following Hardy-Littlewood's result (see, e.g., [9, p. 101]):

$$f\in \mathrm{Lip}\,\alpha\quad (0<\alpha\leq 1)\quad \text{iff}\quad \partial I(x,\xi)/\partial x = \mathcal{O}(\xi^{\alpha-1}),\quad \xi\to 0+.$$

A question which arises is to give an analogous characterization for $\partial P(x,\xi)/\partial x$, $\partial P_p(x,\xi)/\partial x$, $\partial Q(x,\xi)/\partial x$, $\partial W(x,\xi)/\partial x$ and $\partial W_p(x,\xi)/\partial x$, too.

Remark 4.5. Direct and converse approximation results in uniform approximation by linear combinations of Gauss-Weierstrass-type operators obtained in $W(x,\xi)$ by replacing π with $+\infty$ and $-\pi$ with $-\infty$, were given in [10].

Also, the results in [10] are given in terms of the L^p-norm in [11].

Then it would be of interest to obtain the estimates in the present paper by replacing the uniform norm with the L^p-norm, $p>0$.

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Department of Mathematics,
University of Oradea,
5, Armatei Române Str. 3700 Oradea,
Romania