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A CLASS OF DISCRETELY DEFINED POSITIVE LINEAR OPERATORS SATISFYING DEVORE-GOPENGAUZ INEQUALITIES

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1. INTRODUCTION AND HISTORICAL NOTES

An improved form of Jackson's well-known theorem on the approximation of continuous functions given on the interval $I = [0, 1]$ by algebraic polynomials is the pointwise estimate of Theorem 1, which is due to the work of Timan [17] for $k = 1$, Dzjadyk [7], Freud [8] for $k = 2$, and of Brudnyi [1] for $k \geq 3$ (see also [5]).

THEOREM 1. *Let $f \in C(I)$. If $k \in \mathbb{N} = \{1, 2, \ldots\}$, then there is a constant c_k such that for any $n \geq k-1$ we can find an algebraic polynomial $p_n \in \Pi_n$ satisfying*

$$|(f - p_n)(x)| \leq c_k \cdot \omega_k\left(f; \frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}\right), \quad x \in I,$$

where $\omega_k(f, \cdot)$ denotes the k-th order modulus of smoothness of f.

Telyakovskiĭ ([16], $k = 1$), Gopengauz ([13], $k = 2$) and DeVore ([6], $k = 2$) were the first to discover the validity of inequalities of the type given above with the $\frac{1}{n^2}$ term removed, i.e., estimates of the form

$$(1) \qquad |(f - p_n)(x)| \leq c_k \omega_k\left(f; \frac{\sqrt{x(1-x)}}{n}\right), \quad x \in I.$$

Moreover, Yu [18] and Li [14] gave counterexamples showing that an inequality of the type (1) cannot hold for $k \geq 3$.

Recently, Cao and Gonska have given in [2] a simple proof of (1) for the case $k = 2$, at the same time embedding the method of proof into a more general and instructive framework, namely, the mentioned authors have considered the

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Boolean sum of a positive linear operator and the Lagrange polynomial of first order interpolating at the endpoints.

Since the Boolean sum of two positive linear operators is not a positive linear operator, Gonska and Zhou formulated in [12] the following problem:

"Do there exist positive linear operators $L_n : C(I) \to \Pi_n$, such that for all $f \in C(I)$ and all $x \in I$ one has

$$(2) \qquad |L_n(f;x) - f(x)| \le c\omega_k\left(f; \frac{\sqrt{x(1-x)}}{n}\right)$$

with the constant c independent of f, n and x?"

Combining this question with a well-known problem posed by Butzer in 1980 leads to the difficult question whether there exist *discretely defined* positive linear operators satisfying (2).

The first solution to Butzer's problem in its original form was given by Cao and Gonska in 1989 [3]. Furthermore, only recently Gavrea [9] has constructed *non-discrete* positive linear operators satisfying (2).

In the present paper we construct a class of discretely defined positive linear operators satisfying DeVore-Gopengauz inequalities, thus also providing a solution for the stronger form of Butzer's problem as formulated above. Moreover, we shall also investigate the potential of these new operators for simultaneous approximation.

2. CONSTRUCTION OF THE DISCRETE APPROXIMATION OPERATORS

In [9] Gavrea constructed non-discrete positive linear operators satisfying DeVore-Gopengauz inequalities in the following way:

Let $L_n : C(I) \to \Pi_n$ be defined as

$$(1) \qquad (L_n f)(x) = f(0)(1-x)^n + x^n f(1) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-2,k-1}(t) f(t)\,dt,$$

with $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, \dots, n$.

Consider now a polynomial $P_n \in \Pi_n$, $P_n(x) = \sum_{k=0}^{n} a_k x^k$, satisfying the following conditions:

a) $P_n(x) \ge 0$, $x \in I$,

b) $\int_0^1 P_n(x)\,dx = 1$,

c) $P_n'(x) \ge 0$, $x \in I$.

Then the operators $H_{n+2} : C(I) \to \Pi_{n+2}$ are given as

$$(3) \qquad (H_{n+2} f)(x) = \sum_{k=0}^{n} \frac{a_k}{k+1}(L_{k+2} f)(x).$$

In [9] it was proved the following

THEOREM 2. *The operators H_n given in (3) are linear positive operators, and for every function $f \in C(I)$ and every $x \in I$ one has:*

$$(4) \qquad |H_{n+2}(x) - f(x)| \le \frac{9}{4}\omega_2\left(f; \sqrt{x(1-x)}\cdot\sqrt{1 - \int_0^1 x^2 P_n(x)\,dx}\right).$$

In order to turn the estimate (4) into one of the DeVore-Gopengauz type, we have to find polynomials P_n (satisfying the conditions a), b) and c)), such that

$$(5) \qquad 1 - \int_0^1 x^2 P_n(x)\,dx \le \frac{c}{n^2},$$

where c is a constant independent of n.

As one can easily notice, any nonzero polynomial P_n satisfying a) and c) necessarily has the following form:

$$(6) \qquad P_n(x) = \int_0^x Q_{n-1}(t)\,dt + \alpha_n,$$

where $Q_{n-1} \in \Pi_{n-1}$, $Q_{n-1}(x) \ge 0$ for $x \in I$, $\alpha_n \in [0,1)$.

In order to have condition b) also verified by P_n, we have to choose Q_{n-1} such that the equality

$$\int_0^1 (1-x)\,Q_{n-1}(t)\,dx = 1 - \alpha_n$$

is also fulfilled.

On the other hand, from (6) we get

$$1 - \int_0^1 x^2 P_n(x)\,dx =$$

$$= \frac{1}{3}\int_0^1 (1-x)^2 (x+2)\,Q_{n-1}(x)\,dx + \frac{2\alpha_n}{3} \le \int_0^1 (1-x)^2 Q_{n-1}(x)\,dx + \frac{2\alpha_n}{3}.$$

Thus, if there exist two positive constants c_1 and c_2 such that

$$\int_0^1 (1-x)^2 Q_{n-1}(x)\,dx \le \frac{c_1}{n^2} \text{ and } \alpha_n \le \frac{c_2}{n^2},$$

then P_n also satisfies (5).

In the next theorem we shall give the smallest value for the constant c_1 in (7), putting $\alpha_n = 0$ in the sequel.

In order to give a uniform treatment for all the cases in our next theorem, we will introduce now the following notations:

$$\Pi^*_{n-1} = \left\{ P \in \Pi_{n-1} \mid P(x) \ge 0 \text{ for } x \in [0,1] \text{ and } \int_0^1 (1-x)\,P(x)\,dx = 1 \right\},$$

$$s := s(n) = 1 + \left[\frac{n-1}{2}\right],$$

$$d := d(n) = n - 1 - 2\left[\frac{n-1}{2}\right].$$

THEOREM 3. $\inf\limits_{Q_{n-1} \in \Pi^*_{n-1}} \int_0^1 (1-x)^2 Q_{n-1}(x)\,dx = \int_0^1 (1-x)^2 Q^*_{n-1}(x)\,dx = 1 - x_s,$

where $Q^*_{n-1}(x) = \lambda^*_n x^d \left(\frac{J_s^{(1,d)}(x)}{x - x_s}\right)^2$, $\lambda^*_n = (1-x_s^2)\cdot\frac{s+d+1}{s+1}$, *and* x_s *is the largest root of the Jacobi polynomial* $J_s^{(1,d)}$ *relative to the interval* $[0,1]$.

Proof. We shall investigate first the case when n is an odd natural number, that is, $d = 0$ and $n = 2s - 1$. We consider the Gauss quadrature formula

$$(8) \qquad \int_0^1 (1-x)\,f(x)\,dx = \sum_{k=1}^{s} A_k f(x_k) + R(f),$$

where x_k, $k = 1, \dots, s$, are the roots of $J_s^{(1,d)}$. Thus $R(f) = 0$ for $f \in \Pi_{2s-1}$.

Let $Q_{n-1} \in \Pi^*_{n-1}$. We apply the previous quadrature formula for $f(x) = (1-x)\,Q_{n-1}(x)$. It follows that

$$\int_0^1 (1-x)\,(1-x)\,Q_{n-1}(x)\,dx = \sum_{k=1}^{s} A_k (1-x_k) Q_{n-1}(x_k).$$

Because $A_k > 0$, $k = 1, \dots, s$, we get

$$(9) \qquad \int_0^1 (1-x)^2 Q_{n-1}(x)\,dx \ge A_s (1-x_s) Q_{n-1}(x_s).$$

One obtains equality in the latter relation when

$$Q^*_{n-1}(x) = \lambda^*_n \left(\frac{J_s^{(1,0)}(x)}{x - x_s}\right)^2,$$

with

$$\lambda^*_n = \frac{1}{\int_0^1 (1-x)\left(\frac{J_s^{(1,0)}(x)}{x - x_s}\right)^2 dx}.$$

We compute now the integral from (10) by using again the Gauss quadrature formula (8). This gives

$$\int_0^1 (1-x)\left(\frac{J_s^{(1,0)}(x)}{x - x_s}\right)^2 dx = A_s (J_s^{(1,0)\prime}(x_s))^2.$$

The latter equality together with (9) and (10) imply that

$$\inf_{P \in \Pi^*_{n-1}} \int_0^1 (1-x)^2 P(x)\,dx = \int_0^1 (1-x)^2 Q^*_{n-1}(x)\,dx = 1 - x_s,$$

where $Q^*_{n-1}(x) = \lambda^*_n \left(\frac{J_s^{(1,0)}(x)}{x - x_s}\right)^2$, $\lambda^*_n = \left(\frac{J_s^{(1,d)}(x)}{x - x_s}\right)^2$, and λ^*_n is given in (10).

We investigate now the case when n is an even natural number. Thus $n = 2s$, $d = 1$. We shall use Bouzitat's quadrature formula of the first kind (see [10, pp. 101 104]):

$$(11) \qquad \int_0^1 (1-x)\,f(x)\,dx = A_0 f(0) + \sum_{k=1}^{s} A_k f(x_k) + R(f),$$

where x_k are the roots of $J_s^{(1,1)}$ (which is $J_s^{(1,d)}$, and $R(f) = 0$ for $f \in \Pi_{2s}$. The coefficients of Bouzitat's quadrature formula are given by

$$A_0 = \frac{1}{(s+1)(s+2)} \text{ and } A_k = \frac{1 - x_k}{(s+1)(s+2)} \cdot \frac{1}{(J_s^{(1,0)\prime}(x_k))^2}, \quad \textit{for } k = 1, \dots, s.$$

We apply the quadrature formula (11) for $f(x)=(1-x)Q_{n-1}(x)$, with $Q_{n-1}\in\Pi^*_{n-1}$, which gives

(12) $$\int_0^1 (1-x)^2 Q_{n-1}(x)\,dx = A_0 Q_{n-1}(0)+\sum_{k=1}^{s} A_k(1-x_k)\cdot Q_{n-1}(x_k)\ge$$
$$\ge A_s(1-x_s)\,Q_{n-1}(x_s).$$

In order to have equality in the latter relation, $Q_{n-1}(x)$ has to have the form

(13) $$Q_{n-1}(x)=\lambda^*_n\cdot x\left(\frac{J_s^{(1,1)}(x)}{x-x_s}\right)^2,$$
$$\lambda^*_n=\frac{1}{\int_0^1 (1-x)\,x\left(\frac{J_s^{(1,1)}(x)}{x-x_s}\right)^2 dx}.$$

On the other hand, an application of Bouzitat's quadrature formula for the integral in (13) yields

$$\int_0^1 (1-x)\left(\frac{J_s^{(1,1)}(x)}{x-x_s}\right)^2 dx = A_s(J_s'^{(1,1)}(x_s))^2.$$

Using the latter equality and the relations (12) and (13), it follows that

$$\inf_{P\in\Pi^*_{n-1}}\int_0^1 (1-x)^2 P(x)\,dx=\int_0^1 (1-x)^2 Q^*_{n-1}(x)\,dx=1-x_s,$$

with Q^*_{n-1} given as in (13).

Unifying the results for the cases n odd and n even, we can now state the following:

(14) $$\inf_{Q_{n-1}\in\Pi^*_{n-1}}\int_0^1 (1-x)^2 Q_{n-1}(x)\,dx = 1-x_s,\quad n\in\mathbb{N}.$$

Obviously, $\lambda^*_n=\dfrac{1}{\int_0^1 (1-x)\,x^d\left(\frac{J_s^{(1,d)}(x)}{x-x_s}\right)^2 dx}$, $n\in\mathbb{N}$.

In order to compute the integral from the denominator of λ^*_n, we shall use the following Gauss quadrature formula:

$$\int_0^1 (1-x)\,{}^d f(x)\,dx=\sum_{k=1}^{s} A_k f(x_k)+R(f),$$

where x_k, $k=1,\dots,s$ are the roots of $J_s^{(1,d)}$.

We have

$$\int_0^1 (1-x)\,x^d\left(\frac{J_s^{(1,d)}(x)}{x-x_s}\right)^2 dx = A_s(J_s'^{(1,d)}(x_s))^2,$$

where $A_s=\dfrac{s+1}{s+d+1}\cdot\dfrac{1}{(1-x^2)(J_s'^{(1,d)}(x_s))^2}$ (see [10, p. 94]).

Thus we obtain $\lambda^*_n=(1-x_s^2)\dfrac{s+d+1}{s+1}$, which completes the proof. □

The next theorem provides a method which enables us to discretize a linear polynomial operator, such that the discretized operator has the same degree of approximation as the initial (non-discrete) one.

THEOREM 4. *Let* $\mathscr{L}_n : C(I)\to\Pi_n$ *be a positive linear operator of the form*

$$(\mathscr{L}_n f)(x)=\int_0^1 \rho(t)\,K_n(x,t)\,f(t)\,dt,$$

where ρ *is a weight function strictly positive on* (0, 1), *and* $K_n(x,\cdot)\in\Pi_n$ *for every* $x\in I$. *Then:*

(i) $K_n(x,t)\ge 0,\ \forall (x,t)\in I\times I$.

(ii) *The operator* $\mathscr{L}^*_n f : C(I)\to\Pi_n$ *defined by*

$$(\mathscr{L}_n f)(x)=\sum_{k=1}^{n} A_k K_n(x,x_k)\,f(x_k)$$

is a positive linear operator satisfying

$$\mathscr{L}^*_n(e_i)=\mathscr{L}_n(e_i)\quad\text{for } i=0,1,2,$$

whenever A_k *and* $k=1,\dots,n$, *are the coefficients and the nodes of a positive quadrature formula of the form*

(15) $$\int_0^1 \rho(x)\,f(x)\,dx=\sum_{k=1}^{n} A_k f(x_k)+R(f)$$

and having the degree of exactness at least $n+2$.

Proof. (i) For $x \in I$ fixed, let $f_x : I \to \mathbb{R}_+$ be defined as

$$f_x(t) = |K_n(x,t)| - K_n(x,t).$$

Since the operator $\mathcal{L}_n$ is positive, we also have that $(\mathcal{L}_n f_x)(x) \geq 0$. Thus $\int_0^1 \rho(t)\, K_n(x,t)\, f_x(t)\, dt \geq 0$. But $\int_0^1 \rho(t)\, K_n(x,t)\, f_x(t)\, dt = -\frac{1}{2}\int_0^1 \rho(t)\, f_x^2(t)\, dt.$ The last two relations imply now that

$$f_x(t) = 0 \Leftrightarrow |K_n(x,t)| = K_n(x,t) \Leftrightarrow K_n(x,t) \geq 0.$$

(ii) Because $K_n(x,t) \in \Pi_n$ and the quadrature formula (15) has the degree of exactness at least $n+2$, we can write

$$(\mathcal{L}_n e_i)(x) = \int_0^1 \rho(t)\, K_n(x,t)\, t^i\, dt =$$

$$= \sum_{k=0}^{n} A_k K_n(x,x_k) \cdot x_k^i = (\mathcal{L}_n^* e_i)(x),$$

for $i = 0, 1, 2.$

In the sequel, we shall use Theorem 4 in order to construct discretely defined positive linear operators which satisfy DeVore-Gopengauz inequalities.

We shall consider first the operators H_{n+2} defined in (3). One easily notices that these operators can be written as follows:

$$(H_{n+2}f)(x) = f(0)(1-x)^2 \int_0^1 P_n^*(t(1-x))\, dt +$$

$$+ f(1) \cdot x^2 \cdot \int_0^1 P_n^*(tx)\, dt + \int_0^1 K_n(x,t)\, f(t)\, dt,$$

where

$$K_n(x,t) = \sum_{k=0}^{n} \sum_{i=1}^{k+1} a_k p_{k+2,i}(x) \cdot p_{k,i-1}(t)$$

and $a_k,\ k = 0, \ldots, n,$ are the coefficients of the polynomial $P_n^*(x) = \int_0^x Q_{n-1}^*\, dt$, with

$$Q_{n-1}^*(x) = \frac{s+d+1}{s+1}(1-x_s^2)\, x^d \left(\frac{J_s^{(1,d)}(x)}{x - x_s} \right)^2,$$

and s and d are given as in Theorem 3.

Let us consider now a positive quadrature formula (with respect to $r \equiv 1$) of the form

$$\int_0^1 f(x)\, dx = \sum_{k=1}^{n} A_k f(x_k) + R(f),$$

such that $R(f) = 0$ for $f \in \Pi_{n+2}$ (i.e., its degree of exactness is at least $n+2$). Since $K_n(x,t)$ is a polynomial of degree n in t, we have

(17) $$\int_0^1 K_n(x,t) \cdot t^i\, dt = \sum_{k=1}^{n} A_k K_n(x,x_k) \cdot x_k^i, \quad \text{for } i = 0, 1, 2.$$

We construct now the discretely defined operators

(18) $$(H_{n+2}^* f)(x) = (1-x)^2 f(0) \cdot \int_0^1 P_n^*(t(1-x))\, dt +$$

$$+ x^2 f(1) \cdot \int_0^1 P_n^*(xt)\, dt + \sum_{k=1}^{n} A_k \cdot K_n(x,x_k)\, f(x_k).$$

THEOREM 5. *The operators $H_{n+2}^* : C(I) \to \Pi_{n+2}$ given in* (18) *satisfy the following inequality:*

$$|(H_{n+2}^* f)(x) - f(x)| \leq c\,\omega_2\left(f; \frac{\sqrt{x(1-x)}}{n} \right),$$

where the constant c is independent on f, n and x.

Proof. The operator H_{n+2}^* is linear and positive. Thus, using relations (17) and Theorem 2, we get

(19) $$H_{n+2}^* e_i = H_{n+2} e_i = e_i, \quad \text{for } i = 0, 1, \quad \text{and}$$

$$(H_{n+2}^* e_2)(x) = (H_{n+2} e_2)(x) = x^2 + x(1-x)\left(1 - \int_0^1 x^2 P_n^*(x)\, dx \right).$$

Thus relations (19) imply

$$(H_{n+2}^* \Omega_{x,2})(x) = (H_{n+2} \Omega_{x,2})(x),$$

where $\Omega_{x,i}(t) = |t - x|^i,\ i \in \mathbb{N}.$

We shall need now the following result, established by Gonska and Kovacheva in [11]: If L is a positive linear operator defined on $C(I)$ with $Le_0 = e_0$, $Le_1 = e_1$, then for $f \in C(I), x \in I$ and each $h, h \in \left(0, \frac{1}{2}\right]$ we have:

$$(20) \qquad |(Lf)(x) - f(x)| \le \left[\frac{3}{2} + \frac{3}{4}h^{-2} L(e_1 - x)^2; x)\right] \omega_2(f; h).$$

Thus the statement of the theorem holds. □

Our next theorem will prove that one can construct infinitely many operators verifying Theorem 5. We shall denote these operators by $\mathcal{H}^*_{n+2}$.

To that end, we take $Q_{n-1} \in \Pi^*_{n-1}$ of the following form

$$Q_{n-1}(x) = \lambda(Q^*_{n-1}(x) + Q_{n-2}(x)),$$

where Q^*_{n-1} is as in Theorem 3, $Q_{n-2} \in \Pi^*_{n-2}$ such that $Q_{n-2}(x) \ge 0$, $x \in I$, and

$$\lambda = \frac{1}{1 + \int_0^1 (1-x)\, Q_{n-2}(x)\, dx}.$$

We define now $P_n(x) = \int_0^x Q_{n-1}(t)\, dt$ and the operators $\mathcal{H}^*_{n+2}$ given as in (18) with P_n from above. We also denote by $\mathcal{H}_{n+2}$ the continuous operators defined as in (3) and P_n chosen again as above. We shall use the latter operators in Section 3.

THEOREM 6. *The operators $\mathcal{H}^*_{n+2}$ defined above satisfy*

$$|(\mathcal{H}^*_{n+2} f((x) - f(x)| \le c\omega_2\left(f; \frac{\sqrt{x(1-x)}}{n}\right)$$

if $\int_0^1 (1-x)^2 Q_{n-2}(x)\, dx \le \frac{c}{n^2}$.

Proof. It is sufficient to show that there exists a positive constant c independent of f, n and x, such that

$$\int_0^1 (1-x)^2 Q_{n-1}(x)\, dx \le \frac{c}{n^2}.$$

We have:

$$\int_0^1 (1-x)^2 Q_{n-1}(x)\, dx = \lambda\left(1 - x_s + \int_0^1 (1-x)^2 Q_{n-2}(x)\, dx\right) \le \lambda\left(1 - x_s + \frac{c}{n^2}\right).$$

But $0 < \lambda = \dfrac{1}{1 + \int_0^1 (1-x) Q_{n-2}(x)\, dx} < 1$ and $1 - x_s \le \frac{c}{n^2}$ (see [15, p. 288]). Thus $\int_0^1 (1-x)^2 Q_{n-1}(x)\, dx \le \frac{c}{n^2}$.

Remark 7. *We mention here that polynomials Q_{n-2} satisfying the requirement from Theorem 6 exist, for example,*

$$Q_{n-2}(x) = \frac{1}{n^2} P_{n-2}(x),$$

where $P_{n-2} \in \Pi_{n-2}, n \ge 2$ satisfy the condition $0 \le P_{n-2}(x) \le c$, for $x \in I$, and c is a constant independent of n and x.

3. SIMULTANEOUS APPROXIMATION

In this section we shall investigate the potential of the operators $\mathcal{H}_{n+2}$ and $\mathcal{H}^*_{n+2}$ for simultaneous approximation. We will therefore need the next three results given by Cao, Gonska and Kacsó in [4]:

LEMMA 8. *Let $n \ge 2$ and $cn \le m(n) \le \tilde{c}n$. Furthermore, let $L_n: C(I) \to \Pi_{m(n)}$ be a sequence of positive linear operators, satisfying the following conditions:*

(i) $L_n e_0 = e_0$,

(ii) $(L_n \Omega_{x,1})(x) = \mathcal{O}\left(\frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}\right)$.

Then we have for every $f \in C^1(I)$

$$\|(L_n f)'\| \le c \cdot \|f'\|.$$

LEMMA 9. *Let $n \ge 2$, $cn \le m(n) \le \tilde{c}n$, and $L_n: C(I) \to \Pi_{m(n)}$ be a sequence of positive linear operators satisfying the following conditions:*

(i) $L_n e_0 = e_0$,

(ii) $(L_n e_1)(x) - x = \alpha_n x + \beta_n$, where $\alpha_n, \beta_n = \mathcal{O}\left(\frac{1}{n^2}\right)$,

(iii) $(L_n \Omega_{x,2})(x) = \mathcal{O}\left(\frac{x(1-x)}{n^2} + \frac{1}{n^4}\right)$.

Then for every $f \in C^2(I)$, *we have*

$$\|(L_n f)''\| \le c \cdot \|f''\|.$$

LEMMA 10. *Let* $k \in \mathbb{N}$, *and* L_n *be a sequence of linear operators mapping* $C(I)$ *to* $C^k(I)$. *If*

(i) $\lim_{n\to\infty} \|(L_n e_i - e_i)^{(k)}\| = 0$,

(ii) $\|(L_n f)^{(k)}\| \le c \cdot \|f^{(k)}\|$ for all $f \in C^k(I)$,

then for all $f \in C^k(I)$ *the following holds:*

$$\lim_{n\to\infty} \|(L_n f - f)^{(k)}\| = 0.$$

The main result of this section reads now as follows:

THEOREM 11. *Let* $k = 0, 1, 2$. *Then, for every* $f \in C^k(I)$, *we have:*

(i) $\lim_{n\to\infty} \|(\mathcal{H}_{n+2} f - f)^{(k)}\| = 0$,

(ii) $\lim_{n\to\infty} \|(\mathcal{H}^*_{n+2} f - f)^{(k)}\| = 0$.

Proof. Using the Cauchy-Schwartz inequality, one obtains

$$(\mathcal{H}_{n+2}\Omega_{x,1})(x) \le \sqrt{(\mathcal{H}^*_{n+2}\Omega_{x,2})(x)} \le c \cdot \frac{\sqrt{x(1-x)}}{n}.$$

Thus the requirements from Lemmas 8 and 9 are fulfilled. An application of Lemma 10 yields now the statement of our theorem. □

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