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AN INFEASIBLE-INTERIOR-POINT METHOD FOR THE $P_*(\kappa)$-MATRIX LCP

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1. INTRODUCTION

The P_*-matrix linear complementarity problem requires the computation of a vector pair $(x,s)\in\mathbb{R}^{2n}$ satisfying

$$s = Mx + q,\ x^T s = 0,\ (x,s)\geq 0, \tag{1.1}$$

where $q\in\mathbb{R}^n$ and $M\in\mathbb{R}^{n\times n}$ is a P_*-matrix. The class of P_*-matrices was introduced by Kojima et al. [7] and it contains many types of matrices encountered in practical applications. Let κ be a nonnegative number. A matrix M is called a $P_*(\kappa)$-matrix if

$$(1+4\kappa)\sum_{i\in\mathcal{J}_+(x)} x_i[Mx]_i + \sum_{i\in\mathcal{J}_-(x)} x_i[Mx]_i \geq 0,\ \forall x\in\mathbb{R}^n, \tag{1.2}$$

where

$$\mathcal{J}_+(x) = \{i : x_i[Mx]_i > 0\},\ \mathcal{J}_-(x) = \{i : x_i[Mx]_i < 0\},$$

or, equivalently, if

$$x^T Mx \geq -4\kappa \sum_{i\in\mathcal{J}_+(x)} x_i[Mx]_i,\quad \forall x\in\mathbb{R}^n. \tag{1.3}$$

The class of all $P_*(\kappa)$-matrices is denoted by $P_*(\kappa)$, and the class P_* is defined by

$$P_* = \bigcup_{\kappa\geq 0} P_*(\kappa),$$

i.e., M is a P_*-matrix if $M\in P_*(\kappa)$ for some $\kappa\geq 0$.

Obviously, $P_*(0) = PSD$ (the class of positive semi-definite matrices). Every convex quadratic optimization problem can be written as a monotone LCP

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and therefore the P_* LCP generalizes this case. Also, we have $P_* \supset P$, where P is the class of all matrices with positive principal minors. This follows from the fact that a P-matrix M is a $P_*(\kappa)$-matrix for $\kappa = \max\left\{-\dfrac{\lambda_{\min}(M)}{4\gamma(M)}, 0\right\}$, where $\lambda_{\min}(M)$ is the smallest eigenvalue of $(M+M^T)/2$, and $\gamma(M)>0$ is the so-called P-matrix number of M (see [7, Lemma 3.3]).

Most interior-point methods for linear programming have been successfully extended to the monotone LCP. However, there are comparatively fewer results for the P_*-matrix LCP. The potential reduction method given by Kojima et al. [7] solves a $P_*(\kappa)$-matrix LCP in at most $O((\kappa+1)\sqrt{n}L)$ iterations. Nevertheless, no superlinear convergence results have been proved so far for that method. The first algorithm for this new class of LCP having both polynomial complexity and quadratic convergence has been recently proposed by Miao [11]. His method is actually an extension of the Mizuno-Todd-Ye's predictor-corrector algorithm for linear programming [14].

In the above mentioned algorithms it is assumed that the starting point (x^0, s^0) satisfies exactly the linear constraints (i.e., $s^0 = Mx^0 + q$) and lies in the interior of the region defined by the inequality constraints (i.e., the vectors x^0 and s^0 are strictly positive). Such a starting point is called *strictly feasible* or simply *interior*. All the points generated by the algorithm are also strictly feasible, which accounts for the name *interior-point method*. However, in practice it is sometimes very difficult to obtain feasible starting points. Numerical experiments have shown that it is possible to obtain good practical performance by using starting points that lie in the interior of the region defined by the inequality constraints, but do not satisfy the equality constraints (cf. [10]). The points generated by the algorithm will remain in the interior of the region defined by the inequality constraints but, in general, will not satisfy the equality constraints. This property is reflected in the name *infeasible-interior-point algorithm* that has been suggested for such methods. While there is an enormous literature dedicated to the study of interior-point methods, the first results on infeasible-interior-point methods were obtained only a couple of years ago. For a recent survey of the results we refer the reader to [20].

Most of the results on infeasible-interior-point algorithms have been obtained for linear programming. The best computational complexity results obtained so far show that infeasible-interior-point algorithms can solve standard form linear programs with integer data of length L in $O(nL)$ iterations. This complexity is shared by the algorithms proposed in [2], [9], [12], [13], [18] and [19]. The algorithms of [12] and [19] are also quadratically convergent. Ye, Todd and Mizuno [27] have obtained $O(\sqrt{n}L)$-iteration complexity by applying the Mizuno-Todd-Ye algorithm to a homogeneous self-dual reformulation of the original linear programming problems where the original infeasible starting point enters the formulation of the homogeneous self-dual problem itself and becomes a feasible starting point for it. The quadratic convergence of the corresponding iterative process is proved in [22].

The first infeasible-interior-point algorithms for LCP were proposed by Y. Zhang [28] and S. Wright ([23] and [24]) and they had $O(n^2L)$-iteration complexity. Zhang's algorithm was studied for the horizontal linear complementarity problem, while Wright's algorithms, which are also subquadratically convergent, were proposed for the monotone linear complementarity problem. The latter problem is obviously a particular case of the former problem, but, according to the work of Güler [3], horizontal linear complementarity problems can always be reduced to monotone linear complementarity problems, so that the two problems are equivalent. To our knowledge no extensions of infeasible-interior-point methods for more general linear complementarity problems have been proposed so far. However, the computational complexity has been improved. Kojima, Mizuno and Todd [9], mention that the $O(nL)$ infeasible-interior-point algorithms for linear programming considered in that paper can be generalized for linear complementarity problems, but the superlinear convergence of the resulting algorithms has not been yet established. In a recent paper [20], the second author has proposed a new infeasible-interior-point method for monotone linear complementarity problems, whose computational complexity depends on the quality of the starting point. If the starting points are large enough, then the algorithm has $O(nL)$ iteration complexity. If a certain measure of feasibility at the starting point is small enough, then the algorithm has $O(\sqrt{n}L)$ iteration complexity. At each iteration both "feasibility" and "optimality" are reduced exactly at the same rate. The algorithm requires two matrix factorizations and at most three back-solves per iteration and it is quadratically convergent for problems having a strictly complementary solution. Therefore its asymptotic efficiency index in the sense of Ostrowski [17] is $\sqrt{2}$.

In the present paper we extend the algorithm of [20] for solving the $P_*(\kappa)$-matrix LCP. This algorithm also requires two matrix factorizations and at most three back-solves per iteration step. Both "feasibility" and "optimality" are reduced exactly at the same rate as well. It has $O((\kappa+1)^2 nL)$ iteration complexity for a general $P_*(\kappa)$-matrix LCP and arbitrary, sufficiently large, positive starting points. If the starting points are close to being feasible then the computational complexity drops to $O((\kappa+1)\sqrt{n}L)$ iterations. The algorithm is quadratically convergent for problems having a strictly complementary solution. The latter condition is not restrictive because, as shown by Monteiro and Wright [15], such a condition is necessary for superlinear convergence even in the case of the monotone LCP.

The notation used throughout the paper is rather standard: capital letters denote matrices, lower-case letters denote vectors, script capital letters denote sets, and Greek letters denote scalars. All vectors are considered to be column vectors. The components of a vector $u \in \mathbb{R}^n$ will be denoted by $[u]_i$ (and when there is no danger of confusion by u_i), $i = 1, \dots, n$. The relation $u > 0$ is equivalent to $[u]_i > 0$, $i = 1, \dots, n$, while $u \ge 0$ means $[u]_i \ge 0$, $i = 1, \dots, n$. If $u \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, then (u, w) denotes the column vector formed by the components of u and w, i.e., $(u, w) \in \mathbb{R}^{n+m}$, $[(u,w)]_i = [u]_i$ for $1 \le i \le n$ and $[(u,w)]_{n+j} = [w]_j$ for $1 \le i \le m$. We denote $\mathbb{R}^n_+ = \{u \in \mathbb{R}^n : u \ge 0\}$, $\mathbb{R}^n_{++} = \{u \in \mathbb{R}^n : u > 0\}$. If $u \in \mathbb{R}^n$, then $U := \mathrm{Diag}(u)$ denotes the diagonal matrix having the components of u as diagonal entries. The most used norm is the l_2-norm so that we write $\|\cdot\|$ instead of $\|\cdot\|_2$, both for vector norms and for the corresponding matrix norms $\|A\| = \max\{\|Ax\| : \|x\| = 1\}$. Whenever we need other norms like $\|\cdot\|_1$ or $\|\cdot\|_\infty$ we use the corresponding symbol.

2. THE PREDICTOR-CORRECTOR ALGORITHM

We denote the feasible set of the problem (1.1) and its solution set respectively by

$$\mathscr{F} = \{(x,s) \in \mathbb{R}^{2n}_+ : s = Mx + q\} \quad \text{and} \quad \mathscr{F}^* = \{(x^*, s^*) \in \mathscr{F} : x^{*T} s^* = 0\}.$$

Throughout this paper it will be assumed that $\mathscr{F}^*$ is not empty. It is easily seen that $(x^*, s^*) \in \mathscr{F}^*$ if and only if $(x^*, s^*) \ge 0$ is the solution of the following nonlinear system

$$F(x,s) := \begin{pmatrix} Xs \\ Mx - s + q \end{pmatrix} = 0. \tag{2.1}$$

For any given $\varepsilon > 0$ we define the set of ε-*approximate solutions* of (1.1) as

$$\mathscr{F}^*_\varepsilon = \{(x^*, s^*) \in \mathbb{R}^{2n}_+ : x^{*T} s^* \le \varepsilon,\ \|Mx^* - s^* + q\| \le \varepsilon\}.$$

In what follows we shall present an algorithm that finds a point in this set in a finite number of steps. The algorithm depends on two positive constants α and β given by

$$\alpha = \frac{\hat\lambda^2}{\hat\lambda + \sqrt{(1+4\kappa(1+2\kappa))/2}}, \quad \beta = \frac{\hat\lambda}{\hat\lambda + \sqrt{(1+4\kappa(1+2\kappa))/2}}, \tag{2.2}$$

where

$$\hat\lambda = 1 \Big/ \left(\sqrt{1 + \frac{2(1+2\kappa)^2}{1+4\kappa(1+2\kappa)}} + \frac{\sqrt{2}(1+2\kappa)}{\sqrt{1+4\kappa(1+2\kappa)}} \right). \tag{2.3}$$

It follows successively that

$$1/(\sqrt{2}+\sqrt{3}) \le \hat\lambda \le 1/1(1+\sqrt{2}) \tag{2.4a}$$

$$1 - \beta - \kappa\beta^2/n \ge 1 - \beta - 2\kappa\beta^2/n > 0 \tag{2.4b}$$

$$\frac{\sqrt{(1+4\kappa(1+2\kappa))/2}\,\beta^2}{2(1-\beta) - 2\kappa\beta^2/n} \le \frac{\sqrt{(1+4\kappa(1+2\kappa))/2}\,\beta^2}{1-\beta} = \alpha \tag{2.4c}$$

$$\frac{2(1+2\kappa)\beta}{1-\beta} + \frac{(1+4\kappa(1+2\kappa))\beta^2}{2(1-\beta)^2} = 1, \tag{2.4d}$$

$$\beta - \alpha = \Omega(1/(1+\kappa)), \quad \beta - \alpha < 0.5. \tag{2.4e}$$

The starting point of the algorithm can be any pair of strictly positive vectors $(x^0, s^0) \in \mathbb{R}^{2n}_{++}$ that is α-centered in the sense that it belongs to the following set

$$\mathcal{N}_\alpha = \{(x,s) \in \mathbb{R}^{2n}_{++} : \|Xs - \mu e\| \le \alpha\mu\},$$

where, as throughout this paper, we have denoted $\mu = x^T s / n$.

At a typical step of our algorithm we are given a pair $(x,s) \in \mathbb{R}^{2n}_{++}$ and obtain a predictor direction (u, v) by solving the linear system

$$Su + Xv = -Xs, \tag{2.5a}$$

$$Mu - v = r, \tag{2.5b}$$

where r is the residual $r = s - Mx - q$. Clearly, this is just the Newton's direction for the nonlinear system (2.1), whose Jacobian

$$F'(x,s) := \begin{pmatrix} S & X \\ M & -I \end{pmatrix}$$

is nonsingular whenever $x > 0$ and $s > 0$ and M is a P_*-matrix (see [7, Lemma 4.1]). If we take a step length θ along this direction we obtain

$$x(\theta) = x + \theta u, \quad s(\theta) = s + \theta v, \quad \mu(\theta) = x(\theta)^T s(\theta)/n.$$

We define $\overline{\theta}$ as the largest step length for which

(2.6) $$\|X(\theta)\, s(\theta)-(1-\theta)\,\mu e\| \le \beta(1-\theta)\,\mu, \quad \text{for all} \quad 0\le\theta\le\overline{\theta},$$

and consider the predicted pair

(2.7) $$\overline{x}=x+\overline{\theta}u, \quad \overline{s}=s+\overline{\theta}v.$$

We shall see later that these are strictly positive vectors. Therefore the Jacobian $F'(\overline{x},\overline{s})$ is again nonsingular and we can define the corrector direction $(\overline{u},\overline{v})$ as the solution of the following linear system

(2.8a) $$\overline{S}\overline{u}+\overline{X}\overline{v}=(1-\overline{\theta})\,\mu e-\overline{X}\overline{s},$$

(2.8b) $$M\overline{u}-\overline{v}=0.$$

Along this direction we consider the family of pairs

$$\overline{x}(\theta)=\overline{x}+\theta\overline{u}, \quad \overline{s}(\theta)=\overline{s}+\theta\overline{v}.$$

By using (2.8) and the fact that $(\overline{x},\overline{s})>0$, we have

(2.9) $$\overline{X}(\theta)\,\overline{s}(\theta)\ge\theta(1-\overline{\theta})\,\mu e+\theta^2\overline{U}\overline{v}, \quad \text{for} \quad 0\le\theta\le 1.$$

With a unit step length along the corrector direction we obtain a new pair

(2.10) $$\hat{x}=\overline{x}+\overline{u}, \quad \hat{s}=\overline{s}+\overline{v}.$$

It is easily seen that

(2.11) $$\hat{X}\hat{s}=(1-\overline{\theta})\,\mu e+\overline{U}\overline{v}, \quad \hat{\mu}=\frac{1}{n}\hat{x}^T\hat{s}=(1-\overline{\theta})\,\mu\frac{1}{n}\overline{u}^T\overline{v}.$$

If $\overline{u}^T\overline{v}=0$, then we have $\hat{\mu}=(1-\overline{\theta})\,\mu$ and, by defining the new current pair as $(x^+,s^+)=(\hat{x},\hat{s})$, we obtain the same rate of decrease in feasibility and optimality, i.e.,

(2.12) $$r^+=s^+-Mx^+-q=(1-\overline{\theta})\,r, \quad \mu^+=\frac{(x^+)^T s^+}{n}=(1-\overline{\theta})\,\mu.$$

Otherwise a new corrector direction $(\hat{u},\hat{v})$ is obtained by solving the linear system

(2.13a) $$\hat{S}\hat{u}+\hat{X}\hat{v}=\frac{\overline{u}^T\overline{v}}{n}e,$$

(2.13b) $$M\hat{u}-\hat{v}=0.$$

Along this direction we consider the family of pairs

$$\hat{x}(\theta)=\hat{x}+\theta\hat{u}, \quad \hat{s}(\theta)=\hat{s}+\theta\hat{v}.$$

By using (2.8) and (2.13) we obtain

(2.14a) $$\hat{X}(\theta)\,\hat{s}(\theta)=(1-\overline{\theta})\,\mu e+\overline{U}\overline{v}-\theta\frac{\overline{u}^T\overline{v}}{n}e+\theta(\overline{V}\hat{u})+\theta^2\hat{U}\hat{v},$$

(2.14b) $$\hat{\mu}(\theta)=(1-\overline{\theta})\,\mu+\frac{1}{n}\hat{\rho}(\theta),$$

where

(2.15) $$\hat{\rho}(\theta)=\overline{u}^T\overline{v}(1-\theta)+(\overline{v}^T\hat{u}+\overline{u}^T\hat{v})\,\theta+\hat{u}^T\hat{v}\theta^2.$$

Finally, let $\hat{\theta}$ be the smallest positive root of the quadratic equation $\hat{\rho}(\theta)=0$. (In the proof of Theorem 2.4 we shall show that such a $\hat{\theta}$ exists and $0<\hat{\theta}<2$.) The new current pair (x^+,s^+) is defined as

(2.16) $$x^+=\hat{x}+\hat{\theta}\hat{u}, \quad s^+=\hat{s}+\hat{\theta}\hat{v}.$$

It is easily seen that (2.12) holds in this case, too. In order to have a well-defined algorithm, we have to show that $(x^+,s^+)\in\mathcal{N}_\alpha$ so that the above steps can be repeated with (x^+,s^+) instead of (x, s).

Using the technique of [5] (see also [20]), we can compute explicitly the largest number $\overline{\theta}\in[0, 1]$ satisfying (2.6). The result is summarized in the following lemma.

LEMMA 2.1. *If* $(x,s)\in\mathcal{N}_\alpha$, *then the largest number* $\overline{\theta}\in[0, 1]$ *satisfying* (2.6) *is given by*

$$f=\frac{1}{\mu}Xs-e, \quad g=\frac{1}{\mu}Uv,$$

$$\delta=\|g\|, \quad \alpha_0=\beta^2-\|f\|^2, \quad \alpha_1=f^Tg,$$

(2.17a) $$\varphi_1=\alpha_0/(\alpha_1+\sqrt{\alpha_1^2+\alpha_0\delta^2}),$$

(2.17b) $$\overline{\theta}=2/(1+\sqrt{1+4/\varphi_1}),$$

where (u, v) *is the solution of the linear system* (2.5). *Moreover, the pair* $(\overline{x},\overline{s})$ *defined by* (2.7) *satisfies*

(2.18) $$\|\overline{X}\overline{s}-(1-\overline{\theta})\,\mu e\|=\beta(1-\overline{\theta})\,\mu, \quad \overline{x}>0, \quad \overline{s}>0.$$

COROLLARY 2.2. *Let x, s, a, r be four n-dimensional vectors with* $x > 0$ *and* $s > 0$, *and let* $M \in \mathbb{R}^{n\times n}$ *be a* $P_*(k)$ *-matrix. Then the solution* (u, v) *of the linear system*

$$Su + Xv = a, \tag{2.19a}$$

$$Mu - v = b \tag{2.19b}$$

satisfies the following relations:

$$\|Du\| \le \|\tilde{b}\| + \sqrt{\|\tilde{a}\|^2 + \|\tilde{b}\|^2 + 2\kappa\|\tilde{c}\|^2}, \tag{2.20a}$$

$$\|D^{-1}v\| \le \sqrt{\|\tilde{a}\|^2 + \|\tilde{b}\|^2 + 2\kappa\|\tilde{c}\|^2}, \tag{2.20b}$$

$$\|Du\|^2 + \|D^{-1}v\|^2 \le \|\tilde{a}\|^2 + 2\kappa\|\tilde{c}\|^2 + 2\|\tilde{b}\|^2 + 2\|\tilde{b}\|\sqrt{\|\tilde{a}\|^2 + \|\tilde{b}\|^2 + 2\kappa\|\tilde{c}\|^2} \equiv \chi_1^2, \tag{2.20c}$$

$$\|Uv\|^2 \le \frac{1}{8}\|\tilde{a}\|^4 + \frac{1}{4}\chi_1^2(\chi_1^2 - \|\tilde{a}\|^2), \tag{2.20d}$$

where

$$D = X^{-1/2}S^{1/2}; \quad \tilde{a} = (XS)^{-1/2}a, \quad \tilde{b} = D^{-1}b, \quad \tilde{c} = \tilde{a} + \tilde{b}.$$

Proof. By premultiplying (2.19a) and (2.19b) by $(XS)^{-1/2}$ we get

$$\tilde{u} + (\tilde{v} + \tilde{b}) = \tilde{a} + \tilde{b}, \tag{2.21a}$$

$$\tilde{M}\tilde{u} - (\tilde{v} + \tilde{b}) = 0, \tag{2.21b}$$

where $\tilde{u} = Du$, $\tilde{v} = D^{-1}v$ and $\tilde{M} = D^{-1}MD^{-1}$. It is easily seen that $\tilde{M} \in P_*(\kappa)$ (see [7, Theorem 3.5]) and it follows from [Lemma 3.4] that

$$\tilde{u}^T\tilde{M}\tilde{u} = \tilde{u}^T(\tilde{v} + \tilde{b}) \ge -\kappa\|\tilde{a} + \tilde{b}\|^2.$$

Then we have

$$\|\tilde{u}\|^2 + \|\tilde{v}\|^2 = \|\tilde{a}\|^2 - 2\tilde{u}^T\tilde{M}\tilde{u} + 2\tilde{u}^T\tilde{b} \le \|\tilde{a}\|^2 + 2\kappa\|\tilde{a} + \tilde{b}\|^2 + 2\|\tilde{u}\|\,\|\tilde{b}\|. \tag{2.22}$$

Therefore,

$$(\|\tilde{u}\| - \|\tilde{b}\|)^2 + \|\tilde{v}\|^2 \le |\tilde{a}|^2 + \|\tilde{b}\|^2 + 2\kappa\|\tilde{a} + \tilde{b}\|^2. \tag{2.23}$$

Finally, (2.20a)–(2.20b) follow from (2.23) and (2.20c) from (2.20a) and (2.22). It is easily seen from $\|Uv\|^2 = \|\tilde{U}\tilde{v}\|^2$ that (2.20d) follows from Proposition 2.2 of Potra [20]. □

We are now ready to prove that the algorithm described in this section is well defined. For ease of later reference let us first formally define our algorithm.

Algorithm 2.3. *Choose* $(x^0, s^0) \in \mathcal{N}_\alpha$ *and set* $\psi_0 = 1$. For $k = 0, 1, \ldots,$ *do* A1 *through* A7:

A1. *Set* $x = x^k$, $s = s^k$ *and define* $\mu = (x^T s)/\mu$, $r = s - Mx - q$, $\psi = \psi_k$.

A2. *If* $x^T s \le \varepsilon$, *and* $\|r\| \le \varepsilon$, *then report* $(x, s) \in \mathcal{F}_\varepsilon^*$ *and terminate.*

A3. *Find the solution u, v of the linear system* (2.5), *define* $\bar{x}, \bar{s}$ *as in* (2.7), *and set* $\psi_+ = (1 - \bar{\theta})\,\psi$, *where* $\bar{\theta}$ *is given by* (2.17).

A4. *Find the solution* $\bar{u}, \bar{v}$ *of the linear system* (2.8), *and define* $\hat{x}, \hat{s}$ *as in* (2.10).

A5. *If* $\bar{u}^T\bar{v} = 0$, *then set* $x^+ = \hat{x}, s^+ = \hat{s}$ *and go to A7.*

A6. *Find the solution* $\hat{u}, \hat{v}$ *of the linear system* (2.13), *and define* x^+, s^+ *as in* (2.16) *with* $\hat{\theta}$ *being the smallest positive root of* (2.15).

A7. *Set* $x^{k+1} = x^+$, $s^{k+1} = s^+$, $\bar{\theta}_k = \bar{\theta}$, $\mu_k = \mu$, $r^k = r$, $\psi_{k+1} = \psi_+$.

THEOREM 2.4. *For any integer* $k \ge 0$, *Algorithm* 2.3 *defines a pair*

$$(x^k, s^k) \in \mathcal{N}_\alpha, \tag{2.24}$$

and the corresponding residuals satisfy

$$r^k = \psi_k r^0, \quad \mu_k = \psi_k \mu_0, \tag{2.25}$$

where

$$\psi_0 = 1, \quad \psi_k = \prod_{i=0}^{k-1}(1 - \bar{\theta}_i). \tag{2.26}$$

Proof. The proof is by induction. For $k = 0$, (2.24) and (2.25) are clearly satisfied. Suppose that they are satisfied for some $k \ge 0$. As in Algorithm 2.3, we shall omit the index *k*. Therefore we can write

$$(x, s) \in \mathcal{N}_\alpha, \quad r = \psi r^0, \quad \mu = \psi\mu_0.$$

The fact that (2.25) holds for $k+1$ follows immediately from (2.12). By applying Corollary (2.2) to (2.8) and using (2.18), we deduce that

$$\|\bar{U}\bar{v}\| \le \frac{\sqrt{1 + 4\kappa(1 + 2\kappa)\beta^2}}{\sqrt{8}\,(1 - \beta)}(1 - \bar{\theta})\,\mu, \tag{2.27a}$$

$$\|\bar{D}\bar{u}\|^2 + \|\bar{D}^{-1}\bar{v}\|^2 \le \frac{(1 + 2\kappa)\,\beta^2}{(1 - \beta)}(1 - \bar{\theta})\,\mu, \tag{2.27b}$$

where $\overline{D}=\overline{X}^{-1/2}\overline{S}^{1/2}$. On the other hand, from [7, Lemma 3.4] we have

$$\bar{u}^T\bar{v}\ge -\kappa\,\|(\overline{X}\,\overline{S})^{-1}\|\,\|\overline{Xs}-(1-\overline{\theta})\,\mu e\|^2\ge -\frac{\kappa\beta^2}{(1-\beta)}(1-\overline{\theta})\,\mu. \tag{2.28}$$

It is easily seen from (2.11) and (2.28) that

$$\hat{\mu}\ge \frac{1-\beta-\kappa\beta^2/n}{1-\beta}(1-\overline{\theta})\,\mu. \tag{2.29}$$

By using (2.4) and (2.27)–(2.29) together with the fact that $n^{-1}(\bar{u}^T\bar{v})\,e$ is the orthogonal projection of $\overline{Uv}$ on *span*(e), we can write

$$\|\hat{X}\hat{s}-\hat{\mu}e\|=\|\overline{U}\bar{v}-n^{-1}(\bar{u}^T\bar{v})\,e\|\le\|\overline{U}\bar{v}\|\le \tag{2.30}$$

$$\le\frac{\sqrt{1+4\kappa(1+2\kappa)}\,\beta^2}{\sqrt{8}\,(1-\beta)}(1-\overline{\theta})\,\mu\le\frac{\sqrt{(1+4\kappa(1+2\kappa))/2}\,\beta^2}{2\,(1-\beta-\kappa\beta^2/n))}\hat{\mu}\le\alpha\hat{\mu}.$$

By using (2.4), (2.9) and (2.27a), we obtain

$$\overline{X}(\theta)\,\bar{s}(\theta)\ge\theta(1-\overline{\theta})\,(1-\alpha)\,\mu e>0,\quad\text{for}\quad 0<\theta\le 1.$$

Obviously, we have $\bar{x}(0)=\bar{x}>0,\bar{s}(0)=\bar{s}>0$ and $\bar{x}(1)=\hat{x},\bar{s}(1)=\hat{s}$ so that if $\hat{x}>0,\hat{s}>0$ fails, then there must exist a $\theta\in(0,1]$ and an index i such that $[\bar{x}(\theta)]_i\,[\bar{s}(\theta)]_i=0$, which contradicts (2.31). Therefore, we have that $\hat{x}>0,\hat{s}>0$. In the case when $\bar{u}^T\bar{v}=0$ we have $x^+=\hat{x},s^+=\hat{s}$ so that (2.24) holds for $k+1$ as well. It remains to prove that (2.24) is also satisfied when $\bar{u}^T\bar{v}\ne 0$. Applying Corollary 2.2 to (2.13), we deduce that

$$\|\hat{U}\hat{v}\|\le\frac{\sqrt{1+4\kappa(1+2\kappa)}\,\|n^{-1}(\bar{u}^T\bar{v})\,e\|\,\|\overline{U}\bar{v}\|}{\sqrt{8}(1-\beta)\,(1-\overline{\theta})\,\mu}\le\frac{(1+4\kappa(1+2\kappa))\,\beta^2}{8(1-\beta)^2}\frac{|\bar{u}^T\bar{v}|}{\sqrt{n}} \tag{2.32a}$$

$$\|\overline{D}\hat{u}\|^2+\|\overline{D}^{-1}\hat{v}\|^2\le(1+2\kappa)\left\|(\overline{X}\,\overline{S})^{-\frac{1}{2}}\frac{\bar{u}^T\bar{v}}{n}e\right\|^2\le\frac{(1+2k)\,(\bar{u}^T\bar{v})^2}{n(1-\beta)\,(1-\overline{\theta})\,\mu}, \tag{2.32b}$$

which further implies that

$$|\hat{u}^T\hat{v}|\le\sqrt{n}\,\|\hat{U}\,\hat{v}\|\le\frac{(1+4\kappa\,(1+2\kappa))\,\beta^2}{8(1-\beta)^2}|\bar{u}^T\bar{v}|$$

$$|\bar{v}^T\hat{u}+\bar{u}^T\hat{v}|\le[\,\|\overline{D}\bar{u}\|^2+\|\overline{D}^{-1}\bar{v}\|^2\,]^{\frac{1}{2}}\,[\,\|\overline{D}\hat{u}\|^2+\|\overline{D}^{-1}\hat{v}\|^2\,]^{\frac{1}{2}}\le\frac{(1+2\kappa)\,\beta|\bar{u}^T\bar{v}|}{\sqrt{n}\,(1-\beta)}.$$

Similarly, we have

$$\|\overline{U}\hat{v}+\overline{V}\hat{u}\|\le\frac{(1+2\kappa)\,\beta}{1-\beta}\|\overline{U}\bar{v}\|. \tag{2.33}$$

By substituting the above inequalities in (2.15), we get $\hat{\rho}(\theta)\le(\bar{u}^T\bar{v})\,\varrho(\theta)$ if $\bar{u}^T\bar{v}>0$. Otherwise $\hat{\rho}(\theta)\ge(\bar{u}^T\bar{v})\,\varrho(\theta)$ if $\bar{u}^T\bar{v}<0$, where

$$\varrho(\theta)=1-\theta+\frac{(1+2\kappa)\,\beta}{(1-\beta)\,\sqrt{n}}\theta+\frac{(1+4\kappa(1+2\kappa))\,\beta^2}{8(1-\beta)^2}\theta^2.$$

Together with (2.4d), which implies $\varrho(2)<0$ for $n\ge 2$, we have $\hat{\rho}(0)\,\hat{\rho}(2)<0$. Therefore the positive root $\hat{\theta}$ of the quadratic equation $\hat{\rho}(\theta)=0$ satisfies $0<\hat{\theta}<2$. Because $n^{-1}(\bar{u}^T\bar{v})\,e$ is the orthogonal projection of $\overline{U}\,\bar{v}$ onto *Span*(e), and $0<\hat{\theta}<2$, we have

$$\|\overline{U}\,\bar{v}-\hat{\theta}n^{-1}(\bar{u}^T\bar{v})\,e\|\le\|\overline{U}\,\bar{v}\|.$$

By using (2.4), (2.12), (2.14), (2.16), (2.27) and (2.32)–(2.33), we get

$$\|X^+s^+-\mu^+e\|\le 2\|\overline{U}\,\bar{v}\|\le\frac{\sqrt{(1+4\kappa(1+2\kappa))/2}\,\beta^2}{1-\beta}(1-\overline{\theta})\,\mu=\alpha\mu^+.$$

The positivity of x^+ and s^+ can also be proved by continuity based on the following inequality, which is obtained from (2.4), (2.14a), (2.27a) and (2.32)–(2.33):

$$\hat{X}(\theta)\,\hat{s}(\theta)\ge(1-\overline{\theta})\,(1-\alpha)\,\mu e>0,\quad\forall\,0\le\theta\le 2.$$

Finally, it follows that (2.24) is satisfied for $k+1$ and the proof of our theorem is complete. □

3. GLOBAL CONVERGENCE AND POLYNOMIAL COMPLEXITY

In what follows we assume that $\mathcal{F}^*$ is nonempty. Under this assumption we shall prove that Algorithm 2.3, with $\varepsilon=0$, is globally convergent in the sense that $\lim_{k\to 0}\mu_\kappa=0$ and $\lim_{k\to 0}r^k=0$.

LEMMA 3.1. *Assume that $\mathcal{F}^*$ is nonempty. Let $(x^*,s^*)\in\mathcal{F}^*$ and the sequence (x^k,s^k) is generated by Algorithm 2.3. Then*

$$\psi_k((x^k)^Ts^0+(s^k)^Tx^0)\le(1+4\kappa)\,(2+\zeta)\,n\mu_k, \tag{3.1a}$$

$$(1-\psi_k)\,((x^k)^Ts^*+(s^k)^Tx^*)\le(1+4\kappa)\,(1+\psi_k)+(1-\psi_k)\,\zeta)\,n\mu_k, \tag{3.1b}$$

where

$$\zeta = ((x^0)^T s^* + (s^0)^T x^*) / ((x^0)^T s^0). \tag{3.2}$$

Proof. By writing x, s, ψ for x^k, s^k, ψ_k, respectively, and by using (2.25), we have

$$\psi s^0 + (1-\psi) s^* - s = M(\psi x^0 + (1-\psi) x^* - x).$$

By the definition of $P_*(\kappa)$-matrix (cf. (1.3)) and using the fact that $(x^*, s^*) \geq 0$ and $(x, s) > 0$, we have

$$[\psi x^0 + (1-\psi) x^* - x]^T [\psi s^0 + (1-\psi) s^* - s] \geq \tag{3.3}$$

$$\geq -4\kappa(\psi^2 (x^0)^T s^0 + (1-\psi)\,\psi((x^*)^T s^0 + (x^0)^T s^*) + x^T s),$$

where

$$\mathcal{J}_+ = \{i : [\psi x^0 + (1-\psi) x^* - x]_i\, [\psi s^0 + (1-\psi) s^* - s]_i > 0\}.$$

On the other hand, we have

$$[\psi x^0 + (1-\psi) x^* - x]^T [\psi s^0 + (1-\psi) s^* - s] = \tag{3.4}$$

$$= \psi^2 n\mu_0 + (1-\psi)\,\psi((x^0)^T s^* + (s^0)^T x^*) -$$

$$-\psi((x^0)^T s + (s^0)^T x) + x^T s - (1-\psi)(s^T x^* + x^T s^*) + (1-\psi)(x^*)^T s^*$$

The desired inequalities (3.1) follow from (3.3) and (3.4) by using (2.25) and the fact that $(x^*)^T s^* = 0$, $s^T x^* + x^T s^* \geq 0$ and $s^T x^0 + x^T s^0 > 0$. □

From Lemma 3.1 and Corollary 2.2 we shall derive a useful bound for the quantities

$$\delta_k = \|U^k v^k\| / \mu_k, \quad k \geq 0, \tag{3.5}$$

where (u^k, v^k) is obtained at step A3 of Algorithm 2.3. This bound is going to play an important role in our analysis.

LEMMA 3.2. *Let* (u^k, v^k) *be obtained in the k-th iteration at step* A3 *of Algorithm* 2.3 *and let* δ_k *be defined by* (3.5). *Then*

$$\delta_k \leq \delta^* = n\sqrt{.125 + 4(1+\eta)^4 (1+\kappa)^2},$$

where

$$\eta = \sqrt{n}(1+4\kappa)(2+\zeta)\|(S^0)^{-1} r^0\|_\infty / \sqrt{1-\alpha},$$

with ζ *given by* (3.2).

Proof. We omit the index k. Applying Corollary 2.2 to linear system (2.5) and using Lemma 3.1, we can write

$$\|\tilde{a}\| = \|(XS)^{1/2} e\| = \sqrt{n\mu}, \tag{3.6a}$$

$$\|\tilde{b}\| = \|(XS)^{-1/2} Xr\| \leq (1-\alpha)^{-1/2} \mu^{-1/2} \|Xr\| \leq \psi(1-\alpha)^{-1/2} \mu^{-1/2} \|Xr^0\|_1 \leq \tag{3.6b}$$

$$\leq (1-\alpha)^{-1/2} \mu^{-1/2} \|(S^0)^{-1} r^0\|_\infty\, \psi (s^0)^T x \leq \eta\sqrt{n\mu},$$

$$\|\tilde{c}\| \leq \|\tilde{a}\| + \|\tilde{b}\| \leq (1+\eta)\sqrt{n\mu}. \tag{3.6c}$$

Finally, the required inequality follows by substituting (3.6) in (2.20d). □

It is easily seen from (2.17a) that

$$\frac{1}{\varphi_1} \leq (|\alpha_1| + \sqrt{|\alpha_1|^2 + \alpha_0 \delta^2}) / \alpha_0.$$

The right-hand side of the above inequality is increasing in $|\alpha_1|$ and decreasing in α_0. Using the fact that $\alpha_0 \geq \beta^2 - \alpha^2 > 0$ and $|\alpha_1| \leq \|f\|\,\|g\| \leq \alpha\delta$, we obtain

$$1/\varphi_1 \leq \delta / (\beta - \alpha). \tag{3.7}$$

Finally, from Lemma 3.2, (2.17b) and (3.7) we obtain

$$\bar{\theta}_k \geq \theta^* = 2/(1 + \sqrt{1 + 4\delta^* / (\beta - \alpha)}), \quad k \geq 0. \tag{3.8}$$

With the help of (3.8) and Theorem 2.4 we can easily prove the main result of this section, which basically states that Algorithm 2.3 is globally convergent at a linear rate.

THEOREM 3.3. *Suppose that the optimal set* $\mathcal{F}^*$ *is nonempty.*

(i) *If* $\varepsilon = 0$, *then Algorithm 2.3 either finds an optimal solution* $z^* \in \mathcal{F}^*$ *in a finite number of steps or produces an infinite sequence* $z^k = (x^k, s^k, y^k)$ *such that* $\lim_{k\to\infty} (x^k)^T s^k = 0$, *and* $\lim_{k\to\infty} (r^k) = 0$.

(ii) *If* $\varepsilon > 0$, *then Algorithm 2.3 terminates with a* $z \in \mathcal{F}_\varepsilon$ *in at most*

$$K_\varepsilon = \left\lceil \frac{|\ln(\varepsilon/\varepsilon_0)|}{|\ln(1-\theta^*)|} \right\rceil$$

iterations, where $\varepsilon_0 = \max\{\mu_0, \|r^0\|\}$, *and* $\lceil\chi\rceil$ *denotes the smallest integer greater or equal to* χ.

From the above theorem we can obtain polynomial complexity under certain assumptions on the starting point. For the case when the starting point is feasible, or close to being feasible, it is easily seen from (2.4e), (3.8), Lemma 3.2 and Theorem 3.3 that the following corollary holds.

COROLLARY 3.4. *Assume that $\mathscr{F}^*$ is nonempty and that the starting point is chosen such that there is a constant C independent of n satisfying the inequality*

$$(2+\zeta)\|(S^0)^{-1}r^0\|_\infty \le \frac{C}{(1+\kappa)\sqrt{n}}.$$

Then Algorithm 2.3 *terminates in at most* $\widetilde{K}_\varepsilon = O((1+\kappa)\sqrt{n}\ln(\varepsilon_0/\varepsilon))$ *iterations.*

Most of the complexity results on infeasible-interior-point methods are obtained for starting points of the form

$$x^0 = \rho_p e, \quad s^0 = \rho_d e,$$

where ρ_p and ρ_d are sufficiently large positive constants (big M initialization). For such starting points we clearly have $(x^0, s^0) \in \mathcal{N}_\alpha$ and

$$\zeta = \|x^*\|_1/(n\rho_p) + \|s^*\|_1/(n\rho_d), \quad \text{for some} \quad (x^*, s^*) \in \mathscr{F}^*,$$

$$\|(S^0)^{-1}r^0\|_\infty \le 1 + (\rho_p/\rho_d)\|Me\|_\infty + (1/\rho_d)\|q\|_\infty.$$

Therefore, if ρ_p and ρ_d satisfy the inequalities

$$\rho_p \ge n^{-1}\|x^*\|_1, \quad \rho_d \ge \max\{\rho_p\|Me\|_\infty, \|q\|_\infty, n^{-1}\|s^*\|_1\},$$

for some $(x^*, s^*) \in \mathscr{F}^*$, then $\eta \le O((1+\kappa)\sqrt{n})$ and therefore we obtain the following complexity result from (2.4e), (3.8), Lemma 3.2 and Theorem 3.3.

COROLLARY 3.5. *Assume that $\mathscr{F}^*$ is nonempty and that the starting point is chosen of the form* (3.9) *such that* (3.10) *is satisfied for some $(x^*, s^*) \in \mathscr{F}^*$. Then Algorithm* 2.3 *terminates in at most*

$$\widetilde{K}_\varepsilon = O((1+\kappa)^2 n \ln(\varepsilon_0/\varepsilon)) \tag{3.11}$$

iterations.

4. QUADRATIC CONVERGENCE

In the previous section we have proved that Algorithm 2.3 is globally Q-linearly convergent under very general assumptions. Polynomial complexity was

obtained under some additional assumptions on the starting points. In the present section we shall study the asymptotic convergence properties of Algorithm 2.3 under the further assumption that (1.1) has a strictly complementary solution. Let us denote by $\mathscr{F}^c$ the set of all such solutions, i.e.,

$$\mathscr{F}^c = \{(x,s) \in \mathscr{F}^* : [x]_i + [s]_i > 0, i = 1, 2, \dots, n\}.$$

It is well known that there is a unique partition

$$\mathscr{B} \cup \mathscr{N} = \{1, 2, \dots, n\}, \quad \mathscr{B} \cap \mathscr{N} = \emptyset,$$

such that for any $(x,s) \in \mathscr{F}^c$ we have $([x]_i > 0, [s]_i = 0, \forall i \in \mathscr{B})$ and $([x]_i = 0, [s]_i > 0, \forall i \in \mathscr{N})$. This means that the "basic" and "non-basic" variables are invariant for any strictly complementary solution. Let us denote the corresponding partition of M by

$$M = \begin{pmatrix} M_{BB} & M_{BN} \\ M_{NB} & M_{NN} \end{pmatrix}.$$

Also, for any vector $y \in \mathbb{R}^n$ we denote by y_B the vector of components $[y]_i, i \in \mathscr{B}$, and by y_N the vector of components $[y]_i$, $i \in \mathscr{N}$.

LEMMA 4.1. *The iteration sequence $\{(x^k, s^k)\}$ generated by Algorithm* 2.3 *is bounded, i.e.,*

$$0 < [x^k]_i, [s^k]_i \le (1+4\kappa)(z+\zeta)((x^0)^T s^0) \max_{j=1,\dots,n}\left\{\frac{1}{[x^0]_j}, \frac{1}{[s^0]_j}\right\} \equiv \gamma_0. \tag{4.1}$$

Proof. It is easily seen from (2.25) and (3.1a) that $(x^k)^T s^0 + (s^k)^T x^0 \le$ $\le (1+4\kappa)(2+\zeta)(x^0)^T s^0$, which further implies our desired result. □

LEMMA 4.2. *Let $\{z^k \equiv (x^k, s^k)\}$ be generated by Algorithm* 2.3. *For any solution $z^* \equiv (x^*, s^*) \in \mathscr{F}^*$, there is a constant γ_1 such that*

$$|[x^k(1)]_i - [x^*]_i|, \ |[s^k(1)]_i - [s^*]_i| \le \gamma_1 \frac{\|z^k - z^*\|^2}{\mu^k}. \tag{4.2}$$

Proof. For any $z^* \equiv (x^*, s^*) \in \mathscr{F}^*$, using (2.5) we have

$$\begin{pmatrix} S^k & X^k \\ M & -I \end{pmatrix}\begin{pmatrix} x^k(1) - x^* \\ s^k(1) - s^* \end{pmatrix} = \begin{pmatrix} (X^k - X^*)(s^k - s^*) \\ 0 \end{pmatrix}.$$

Applying Corollary 2.2 to the above linear system, we have

$$(4.3)\qquad \|D^k(x^k(1)-x^*\|\le\sqrt{1-2\kappa}\,\|(X^kS^k)^{-\frac{1}{2}}(X^k-X^*)(s^k-s^*)\|\le$$

$$\le\frac{\sqrt{1+2\kappa}}{\sqrt{1-\alpha}}\,\frac{\|z^k-z^*\|^4}{\sqrt{\mu^k}},$$

where $D^k=(X^k)^{-\frac{1}{2}}(S^k)^{\frac{1}{2}}\equiv Diag(d)$. Thus

$$|[x^k(1)]_i-[x^*]_i|=[d]_i^{-1}|[d]_i([x^k(1)]_i-[x^*]_i)|\le$$

$$\le\frac{[x^k]_i}{([x^k]_i[s^k]_i)^{\frac{1}{2}}}\|D^k(x^k(1)-x^*)\|\le$$

$$\le\gamma_1\frac{\|z^k-z^*\|^2}{\mu^k},\quad\text{with}\quad\gamma_1\equiv\frac{\sqrt{(1+2\kappa)\,\gamma_0}}{1-\alpha}.$$

The inequality involving s can be obtained similarly. □

LEMMA 4.3. *Let $\mathscr{F}^c\ne\emptyset$. Then there is a constant γ_2 such that*

$$[x^k]_i\le\gamma_2(x^k)^Ts^k,\quad\forall i\in\mathscr{N},\qquad [s^k]_i\le\gamma_2(x^k)^Ts^k,\quad\forall i\in\mathscr{B}.$$

Proof. Let $(x^*,s^*)\in\mathscr{F}^c$. It is easily seen from (3.1b) and the fact that $\psi_k\le\psi_1$, $k\ge1$ that

$$(x^k)^Ts^*+(s^k)^Tx^*\le(1+4\kappa)\,((1+\psi_1)/(1-\psi_1)+\zeta)\,n\mu_k\le$$

$$\le(1+4\kappa)\,((2-\bar\theta_0)/\bar\theta_0+\zeta)\,(x^k)^Ts^k.$$

Therefore,

$$[x^k]_i\le\frac{(1+4\kappa)\,((2-\bar\theta_0)/\bar\theta_0+\zeta)}{[s^*]_i}(x^k)^Ts^k,\quad\forall i\in\mathscr{N},$$

and

$$[s^k]_i\le\frac{(1+4\kappa)\,((2-\bar\theta_0)/\bar\theta_0+\zeta)}{[x^*]_i}(x^k)^Ts^k,\quad\forall i\in\mathscr{B}.$$

Hence the desired result holds with

$$\gamma_2=(1+4\kappa)\,((2-\bar\theta_0)/\bar\theta_0+\zeta)\max\left\{\max_{i\in\mathscr{B}}\frac{1}{[x^*]_i},\ \max_{i\in\mathscr{N}}\frac{1}{[s^*]_i}\right\}.\ \square$$

LEMMA 4.4. *Suppose that $\mathscr{F}^c\ne\emptyset$. Let $\{z^k\equiv(x^k,s^k)\}$ be generated by Algorithm 2.3. Then there is a constant γ_3 such that for each k there is a solution $z_*^k\in\mathscr{F}^*$ such that*

$$(4.4)\qquad \|z^k-z_*^k\|\le\gamma_3\mu^k.$$

Proof. Consider the following equality-inequality linear system:

$$(4.5)\qquad\begin{array}{llll} M_{BB}x_B & & & =-q_B\\ M_{NB}x_B & & -s_N & =-q_N\\ & x_N & & =0\\ & s_B & & =0\\ & x_B, & s_N & \ge0. \end{array}$$

Under the assumption that $\mathscr{F}^c\ne\emptyset$, (4.5) has a solution and the solution set of (4.5) is $\mathscr{F}^*$. By Hoffman's lemma [4], for any z^k, there is a constant γ_4, independent of k and $z_*^k\in\mathscr{F}^*$ such that

$$(4.6)\qquad \|z^k-z_*^k\|\le\gamma_4\|(M_{BB}x_B^k+q_B,\ M_{NB}x_B^k+q_N-s_N,\ x_N^k,s_B^k)\|\le$$

$$\le\gamma_4\|(-M_{BN}x_N^k+s_B^k,\ -M_{NN}x_N^k,\ x_N^k,s_B^k)\|+\gamma_4\|r^k\|.$$

Moreover,

$$\|r^k\|=\|\psi r^0\|=\frac{\|r^0\|}{\mu_0}(\psi\mu_0)=\frac{\|r^0\|}{\mu_0}\mu.$$

Finally, (4.4) follows from Lemma 4.3 and (4.6)–(4.7). □

LEMMA 4.5 *Let $\{z^k\equiv(x^k,s^k)\}$ be generated by Algorithm 2.3. Then*

$$(4.8)\qquad |[u]_i|\le\gamma_5\mu^k,\quad |[v]_i|\le\gamma_5\mu^k,\quad\textit{with}\ \gamma_5\equiv(1+\gamma_1\gamma_3)\gamma_3.$$

Proof. Let $z_*^k=(x_*^k,s_*^k)\in\mathscr{F}^*$ satisfy (4.4). It follows from Lemma 4.2 and Lemma 4.4 that

$$|[u]_i|\le|[x^k]_i+[u]_i-[x_*^k]_i|+|[x_*^k]_i-[x^k]_i|=|[x^k(1)]_i-[x_*^k]_i|+|[x_*^k]_i-[x^k]_i|\le$$

$$\le\gamma_1\gamma_3\|z^k-z_*^k\|+\|z^k-z_*^k\|\le(1+\gamma_1\gamma_3)\gamma_3\mu^k\equiv\gamma_5\mu^k.$$

Similarly we can obtain the inequality for $|[v]_i|$. □

We end the paper by stating and proving our quadratic convergence result.

THEOREM 4.6. *If the linear complementarity problem* (1.1) *has a strictly complementarity solution, then there are two constants* γ *and* $\bar{\gamma}$ *independent of* k *such that the points produced by Algorithm 2.3 satisfy*

$$\mu_{k+1} \le \gamma\mu_k^2, \quad \|r^{k+1}\| \le \bar{\gamma}\|r^k\|^2, \quad k \ge 1. \tag{4.9}$$

Proof. From (2.17), (3.7), (3.5) and (4.8) it follows that

$$\theta_k \ge 1 - \delta/(\beta-\alpha) \ge 1 - \gamma\mu$$

with $\gamma = \sqrt{n}\gamma_5^2/(\beta-\alpha)$. From (4.7) we see that (4.9) holds with $\bar{\gamma} = \gamma\mu^0/\|r^0\|$. □

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