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AN IMPROVED BOUNDARY ELEMENT METHOD FOR THE 2D LIFTING AIRFOIL PROBLEM

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Abstract. Using an inverse Kármán-Trefftz transformation, the problem of the lifting flow around an airfoil with a sharp trailing-edge is reduced to the problem of the flow past a smooth curve which may be solved without any difficulty by means of the boundary element method. This new approach gives the possibility to extend the application of the boundary element method to thin profiles and to profiles with cusped trailing-edge. A comparison between analytical and calculated values of the pressure coefficient in the same control points for some airfoils shows a very good agreement.

1. INTRODUCTION

The boundary element method is a numerical approach to solve boundary value problems obtained by transformation of the partial differential equations into integral equations. These integral equations can then be solved numerically by discretizing the boundary of the domain only. This reduction of dimensionality is a great advantage over domain-type approaches (finite difference method, finite element method) because it is more economical from the computational point of view.

The study of the incompressible potential flow past an arbitrary airfoil by the boundary element method is already a classical problem presented in textbooks dedicated to computational fluid mechanics [1], [2]. Generally the investigation of the incompressible flow past a smooth obstacle is a simple problem. This problem becomes more complicated for the obstacles (airfoils) with a sharp trailing-edge. In this case, for determining the circulation one has to use the Kutta-Joukovsky condition which states that "the flow leaves the sharp trailing-edge of an airfoil smoothly and the velocity there is finite".

Many attempts were made for transforming the Kutta-Joukovsky condition into a numerical relation that may be used in the frame of boundary element

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method. Among them we mention the papers [3], [4], in which one imposes the zero value to the tangential velocity in the trailing-edge. In other papers [1], [5], one imposes the same value of the velocity on both the upper and lower panels of the trailing-edge.

The application of the boundary element method is very difficult for the airfoils with cusped trailing edge because of the very tight placing of the collocation points in the vicinity of the trailing-edge. The same difficulty arises for thin airfoils. The limit case of an airfoil which is an open curve (the airfoil reduced to its skeleton) is quite impossible to be investigated using the boundary element method because of the coincidence of the panels from the upper and lower surfaces.

However, herein we develop a new method which allows us to bypass these difficulties and to apply the boundary element method even to the profiles for which the method eas considered not to be applicable.

The first step of this method is to map conformally the airfoil with a sharp edge onto an arbitrary closed smooth curve by means of an inverse Kármán-Trefftz (or Joukovsky) transformation. We may calculate using the boundary element method the tangential velocity over the smooth curve, and then, using the Kármán-Trefftz (or Joukovsky) transformation we may find out the tangential velocity on the original aitfoil. As we shall see, the Kutta-Joukovsky condition is satisfied by imposing the velocity to vanish in the point on the smooth obstacle corresponding to the trailing-edge.

The idea of using the Kármán-Trefftz transformation for mapping the airfoil onto a smooth curve belongs to Theodorsen [6]. He observed that by means of an appropriate inverse Kármán-Trefftz transformation, many airfoils are mapped onto pseudo-circles. Using a standard numerical method [7] it is possible to determine the conformal mapping of a pseudo-circle onto a circle and afterwards, by means of the conformal mapping method, the tangential velocity on the airfoil.

In comparison with Theodorsen's method, the present method is simpler and more general (as we may see in Fig. 2, it may happen to obtain by means of the inverse Kármán-Trefftz transform a smooth curve that differs very much from a circle).

In the present paper we investigate by means of the boundary element method (using the integral equation for the steam function) the plane, steady, incompressible, potential flow (uniform at infinity) in the presence of an arbitrary profile.

The comparison between the theoretical values and the calculated values of the pressure coefficients obtained for a Von Mises profile, a Kármán-Trefftz profile and a circular arc show a very good agreement.

2. THE KÁRMÁN-TREFFTZ TRANSFORMATION

We consider in the Z-plane a smooth closed curve $\mathscr{L}$, the point Z_0 of $\mathscr{L}$ and the point Z_1 in the interior of the domain bounded by $\mathscr{L}$.

The Kármán-Trefftz transformation

(2.1) $$z = k\frac{Z_0-Z_1}{2}\,\frac{(Z-Z_1)^k+(Z-Z_0)^k}{(Z-Z_1)^k-(Z-Z_0)^k}+k\frac{Z_0+Z_1}{2};\quad 1<k<2,$$

which is equivalent to

(2.2) $$\frac{z-kZ_0}{z-kZ_1}=\frac{(Z-Z_0)^k}{(Z-Z_1)^k}$$

maps conformally the exterior of $\mathscr{L}$ onto the exterior of a curve ℓ in the z-plane.

The curve l has an angular point $z_0 = kZ_0$ (the trailing-edge) and the trailing-edgé angle is

(2.3) $$\tau = \pi\,(2-k).$$

The point $z_1 = kZ_1$ lies in the interior of l.

For $k = 2$, $Z_0 = R > 0$ and $Z_1 = -R$ the Kármán-Trefftz transformation becomes the Joukovsky transformation

(2.4) $$z = Z + \frac{R^2}{Z}$$

which transforms the exterior of a smooth curve containing the point $Z = R$ onto the exterior of a profile having a cusped trailing-edge in $z = 2R$ (we suppose that $Z = -R$ is in the interior of the domain bounded by the smooth curve).

Conversely, the exterior of a profile having the trailing edge z_0 and the trailing edge angle $\pi(2-k)$ is transformed conformally by the inverse Kármán-Trefftz transformation

(2.5) $$Z = \frac{z_0-z_1}{2k}\,\frac{(z-z_1)^{1/k}+(z-z_0)^{1/k}}{(z-z_1)^{1/k}-(z-z_0)^{1/k}}$$

onto the exterior of a smooth curve (we suppose that z_1 is in the interior of the profile).

If $k = 2$ the profile has a cusped trailing edge. Setting $z_0 = 2R$ and $z_1 = -2R$ we obtain the inverse Joukovsky transformation

(2.6) $$Z = \frac{z+(z^2-4R^2)^{1/2}}{2}.$$

As we could see, the Joukovsky transformation may be regarded as a particular Kármán-Trefftz transformation. We can easily check that for the Kármán-Trefftz transformation, (2.7) $$\lim_{Z\to\infty}\frac{dz}{dZ}=1.$$

From (2.2) putting $z_0 = kZ_0$, we get

(2.8) $$z-z_0=(Z-Z_0)^k\Phi(Z);\quad \Phi(Z_0)\neq 0,$$

whence

(2.9) $$\frac{dz}{dZ}=(Z-Z_0)^{k-1}\chi(Z);\quad \chi(Z_0)\neq 0.$$

3. THE METHOD OF CONFORMAL MAPPING FOR 2D INCOMPRESSIBLE FLOW

By means of the inverse Kármán-Trefftz (or Joukovsky) transformation we map conformally the exterior of an airfoil with a sharp trailing-edge in the z-plane onto the exterior of a smooth curve in the Z-plane. We denote by z_0 the trailing-edge and by Z_0 the corresponding point on the smooth curve. The complex velocity in the airfoil plane is

(3.1) $$\frac{df}{dz}=v_1-iv_2,$$

where f is the complex potential of an uniform at infinity flow deflected by the airfoil. At infinity f has the following development

(3.2) $$f(z)=V_\infty e^{-i\alpha}z+\frac{\Gamma}{2\pi i}\ln z+a_0+a_1z^{-1}+a_2z^{-2}+\ldots$$

α is the incidence angle, $V_\infty e^{-i\alpha}$ is the complex velocity at infinity and Γ is the circulation around the airfoil.

The function

(3.3) $$f(Z)=f(z(Z))$$

is the complex potential of the flow past the smooth obstacle in the Z-plane and has the development at infinity

(3.4) $$f(Z)=V_\infty e^{-i\alpha}Z+\frac{\Gamma}{2\pi i}\ln Z+A_1Z^{-1}+A_2Z^{-2}+\ldots$$

By means of a formal manipulation, equation (3.1) may be rewritten

(3.5) $$\frac{df}{dz}=\frac{df}{dz}\frac{dZ}{dz}=v_1-iv_2.$$

Since according to Kutta-Joukovsky condition the velocity is finite at the trailing-edge, from (2.9) and (3.5) we get

(3.6) $$\lim_{Z\to Z_0}\frac{df}{dZ}=\lim_{Z\to Z_0}(Z-Z_0)^{k-1}\chi(Z)(v_1-iv_2)=0.$$

Hence on the smooth obstacle in the Z-plane the velocity vanishes in Z_0.

4. THE APPLICATION OF THE BOUNDARY ELEMENT METHOD FOR SIMULATING THE FLOW PAST THE SMOOTH OBSTACLE

As we deduce from (3.4) the flow past the smooth obstacle in the Z-plane is uniform at infinity and the complex velocity there is $V_\infty e^{-i\alpha}$. We get for the stream function $\psi=\operatorname{Im} f$, the expansion at infinity

(4.1) $$\psi(X,Y)=V_\infty(Y\cos\alpha-X\sin\alpha)-\frac{\Gamma}{2\pi}\ln R+O(R^{-1}),$$

where $R=(X^2+Y^2)^{1/2}$.

Introducing the perturbation stream function

(4.2) $$\Psi(X,Y)=\psi(X,Y)-V_\infty(Y\cos\alpha-X\sin\alpha)$$

we deduce the behaviour of $\Psi(X,Y)$ at infinity:

(4.3) $$\Psi(X,Y)=\frac{\Gamma}{2\pi}\ln R+O(R^{-1}).$$

Taking into account that Ψ is a harmonic function, we have for $(X, Y)\in\mathscr{L}$ the integral representation (valid for functions which have at infinity the behaviour given by (4.3)):

(4.4) $$\pi\Psi(X,Y)=\int_{\mathscr{L}}\left(\frac{\partial\Psi}{\partial n}(\xi,\eta)\ln\frac{1}{r}-\Psi(\xi,\eta)\frac{\partial}{\partial n}\ln\frac{1}{r}\right)ds.$$

(ξ,η) represents the current point on the smooth curve $\mathscr{L}$, ds is the arc element on $\mathscr{L}$ and $\partial/\partial n$ is the inward normal derivative on $\mathscr{L}$. We also denoted by $r=\left((X-\xi)^2+(Y-\eta)^2\right)^{1/2}$.

On the streamline constituted by the airfoil we have

(4.5) $$\psi_{|\mathscr{L}}=q=\text{const.}$$

whence, by virtue of (4.2) it follows

(4.6) $$\Psi_{|\mathscr{L}}=q-V_\infty(Y\cos\alpha-X\sin\alpha)$$

Introducing (4.6) in (4.4) and taking into account that

$$\int_{\mathcal{L}} \frac{\partial}{\partial n}\ln\frac{1}{r}\,ds = -\pi \quad \text{for } (X,Y)\in\mathcal{L} \tag{4.7}$$

we get

$$-2\pi q + \int_{\mathcal{L}} \frac{\partial\Psi}{\partial n}(\xi,\eta)\ln\frac{1}{r}\,ds = -\pi V_\infty(Y\cos\alpha - X\sin\alpha) - V_\infty\int_{\mathcal{L}}(\eta\cos\alpha-\xi\sin\alpha)\frac{\partial}{\partial n}\ln\frac{1}{r}\,ds. \tag{4.8}$$

$\frac{\partial\Psi}{\partial n}$ is the tangential perturbation velocity on the smooth curve $\mathcal{L}$. From (3.6) and (4.3) it follows

$$\frac{\partial\Psi}{\partial n}(X_0,Y_0) = -V_\infty\left(n_{Y_0}\cos\alpha - n_{X_0}\sin\alpha\right). \tag{4.9}$$

The existence and unicity of the solution $(\partial\Psi/\partial n, q)$ of the first kind integral equation (4.8), (4.9) is stated in [8].

Approximating the smooth curve $\mathcal{L}$ by a contiguous polygonal line consisting of N panels $\mathcal{L}_j, j = 0, 1, \ldots, N-1$, and the function $\partial\Psi/\partial n$ on $\mathcal{L}_j$ with its value in the midpoint (X_j, Y_j) of $\mathcal{L}_j$ we obtain from (4.8) and (4.9) the algebric system:

$$-2\pi q + \sum_{i=0}^{N-1}\frac{\partial\Psi}{\partial n}(X_i,Y_i)\int_{\mathcal{L}_i}\ln\frac{1}{r_j}\,ds = -\pi V_\infty(Y_j\cos\alpha - X_j\sin\alpha) - V_\infty\int_{\mathcal{L}}(\eta\cos\alpha-\xi\sin\alpha)\frac{\partial}{\partial n}\ln\frac{1}{r_j}\,ds;\quad j=\overline{0,N-1} \tag{4.10}$$

of N equations with N unknowns q and $\frac{\partial\Psi}{\partial n}(X_i,Y_i), i=\overline{1,N-1}$. In (4.10) we denoted by $r_j = \left(\left(X_j-\xi\right)^2+\left(Y_j-\eta\right)^2\right)^{1/2}$.

The coefficients

$$G_{ij} = \int_{\mathcal{L}_i}\ln\frac{1}{r_j}\,ds \tag{4.11}$$

may be computed numerically if $i \neq j$. For N large enough the approximation

$$\int_{\mathcal{L}_i}\ln\frac{1}{r_j}\,ds = -2l_i\ln\left[\left(X_j-X_i\right)^2+\left(Y_j-Y_i\right)^2\right] \tag{4.12}$$

gives good results (we denoted by l_i the length of $\mathcal{L}_i$).

If $i = j$ we obtain analytically [9]:

$$G_{ii} = l_i\left(1+\ln\frac{2}{l_i}\right). \tag{4.13}$$

For the term in the right part of equation (4.10) we may use the quadrature formula

$$-\pi V_\infty(Y_j\cos\alpha - X_j\sin\alpha) - V_\infty\int_{\mathcal{L}}(\eta\cos\alpha-\xi\sin\alpha)\frac{\partial}{\partial n}\ln\frac{1}{r_j}\,ds = \sum_{i=0}^{N-1}H_{ij}(Y_j\cos\alpha - X_j\sin\alpha), \tag{4.14}$$

where

$$H_{ij} = V_\infty l_i\frac{(X_i-X_j)n_{X_i}+(Y_i-Y_j)n_{Y_i}}{(X_i-X_j)^2+(Y_i-Y_j)^2}, \tag{4.15}$$

$$H_{ii} = -\pi V_\infty. \tag{4.16}$$

The convergence of the solution of the algebric system (4.10) to the exact values of $\partial\Psi/\partial n$ in the control points when N goes to infinity is dicussed in [10].

Using the solutions of (4.10), by means of the relation

$$\frac{\partial\psi}{\partial n}(X,Y) = \frac{\partial\Psi}{\partial n}(X,Y) + V_\infty\left(n_Y\cos\alpha - n_X\sin\alpha\right) \tag{4.17}$$

obtained from (4.2), we may calculate the tangential velocity in the control points (X_i, Y_i), $i=\overline{0,N-1}$ on the smooth obstacle.

5. THE LIFTING FLOW PAST AIRFOILS WITH SHARP TRAILING-EDGE. NUMERICAL RESULTS

We denote by s the arc length on the airfoil and by S the arc length on the corresponding smooth curve. The tangential velocity on the airfoil is

$$v\cdot s = \frac{\partial\varphi}{\partial s}\Big|_l = \frac{\partial\varphi}{\partial S}\Big|_{\mathcal{L}}\cdot\frac{dS}{ds}\Big|_l, \tag{5.1}$$

where φ is the real potential of the velocity. From the Cauchy-Riemann equation

$$\frac{\partial\psi}{\partial n}\Big|_{\mathcal{L}} = \frac{\partial\varphi}{\partial s}\Big|_{\mathcal{L}} \tag{5.2}$$

and from the relation

$$\frac{dS}{ds} = \left|\frac{dZ}{dz}\right| \tag{5.3}$$

we deduce the values of the tangential velocity on the airfoil in the control points (x_i, y_i) corresponding by the Kármán-Trefftz (or Joukovsky) transformation to the control points (X_i, Y_i):

$$v \cdot s = \frac{\partial \psi}{\partial n}\Big|_{\mathcal{V}} \cdot \left|\frac{dZ}{dz}\right|. \tag{5.4}$$

In the present paper we shall study the flow past three different airfoils. The first is a Von Mises airfoil with a cusped trailing-edge. Using the transformation

$$z = \zeta + \frac{21}{25}\zeta^{-1} + \frac{2}{25}\zeta^{-2} \tag{5.5}$$

we map conformally the exterior of the circle

$$\zeta = \frac{8}{5}\cdot\frac{e^{i\theta}}{\cos\theta_0} - \frac{8}{15} - \frac{8i}{5}\operatorname{ctn}\theta_0, \quad \theta_0 = -\frac{\pi}{20}, \quad \theta \in [0, 2\pi] \tag{5.6}$$

onto the exterior of a Von Mises profile (Fig. 1, solid line).

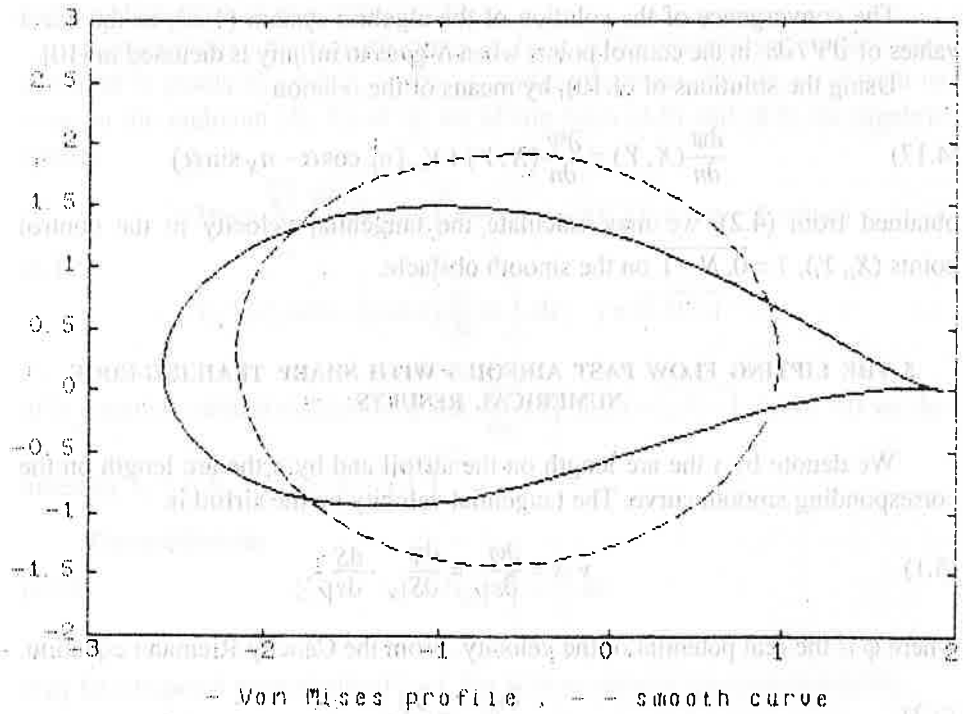

Fig. 1. – Von Mises profile and corresponding curve.

By means of the inverse Joukovsky transformation (2.6) with $R = 0.96$ we map the exterior of the Von Mises profile onto the exterior of a smooth curve (Fig. 1, dashed line).

The second airfoil we are taking into consideration is a Kármán-Trefftz profile (Fig. 2, solid line) obtained from the circle

$$\zeta = \frac{6}{5}\cdot\frac{e^{i\theta}}{\cos\theta_0} - 0.2 - \frac{6i}{5}\operatorname{ctn}\theta_0, \quad \theta_0 = -\frac{\pi}{20}, \quad \theta \in [0, 2\pi] \tag{5.7}$$

by means of the conformal mapping

$$z = 2k\frac{(\zeta+1)^k + (\zeta-1)^k}{(\zeta+1)^k - (\zeta-1)^k} - k, \quad k = 1.8. \tag{5.8}$$

Using the inverse Karman-Trefftz transformation

$$Z = \frac{(z+k)^{1/k} + (z-k)^{1/k}}{(z+k)^{1/k} - (z-k)^{1/k}} \tag{5.9}$$

we map the Kármán-Trefftz profile onto a smooth curve (Fig. 2, dashed line). Although it was possible to choose as smooth curves corresponding to Von Mises and Kármán-Trefftz profiles the circles from the ζ-plane, we prefered to take into consideration the curves in the Z-plane obtained using the relations (2.6) and (5.9) in order to simulate the general case.

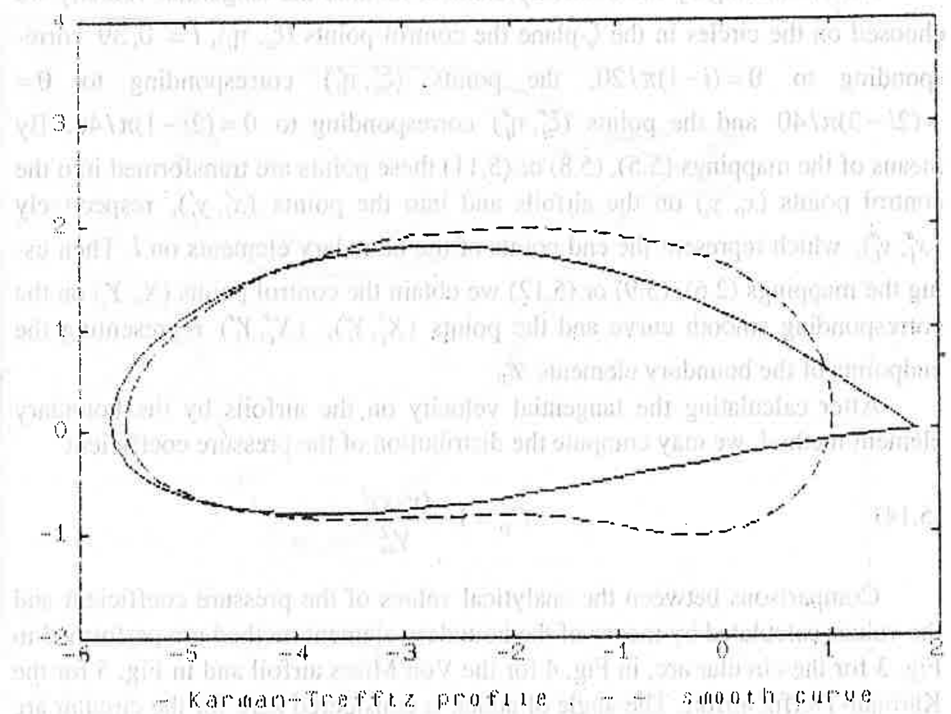

Fig. 2. – Kármán-Trefftz profile and corresponding curve.

The third airfoil we are taking into consideration is the circular arc, obtained from the circle

$$\zeta = \frac{e^{i\theta}}{\cos\theta_0} - i\,\mathrm{ctn}\,\theta_0, \quad \theta_0 = -\frac{\pi}{20}, \quad \theta \in [0, 2\pi] \tag{5.10}$$

by means of the Joukovsky transformation

$$z = \zeta + \zeta^{-1}. \tag{5.11}$$

Using the inverse Joukovsky transformation,

$$Z = \frac{z + (z^2 - 4)^{1/2}}{2} \tag{5.12}$$

we map the exterior of the circular arc onto the exterior of the circle (5:10).

The tangential velocity over the airfoils above mentioned may be obtained analytically by means of the relation

$$v \cdot s = 2V_\infty\left[\sin(\theta_0 - \alpha) - \sin(\theta - \alpha)\right] \cdot \left|\frac{d\zeta}{dz}\right|.$$

For calculating by the boundary element method the tangential velocity we choosed on the circles in the ζ-plane the control points (ξ_i, η_i), $i = \overline{0, 39}$ corresponding to $\theta = (i-1)\pi/20$, the points (ξ_i', η_i') corresponding to $\theta = (2i-3)\pi/40$ and the points (ξ_i'', η_i'') corresponding to $\theta = (2i-1)\pi/40$. By means of the mappings (5.5), (5.8) or (5.11) these points are transformed into the control points (x_i, y_i) on the airfoils and into the points (x_i', y_i'), respectively (x_i'', y_i''), which represent the end points of the boundary elements on l. Then using the mappings (2.6), (5.9) or (5.12) we obtain the control points (X_i, Y_i) on the corresponding smooth curve and the points (X_i', Y_i'), (X_i'', Y_i'') representing the endpoints of the boundary elements $\mathscr{L}_i$.

After calculating the tangential velocity on the airfoils by the boundary element method, we may compute the distribution of the pressure coefficient

$$C_p = 1 - \frac{(v \cdot s)^2}{V_\infty^2}. \tag{5.14}$$

Comparisons between the analytical values of the pressure coefficient and the values calculated by means of the boundary element method are performed in Fig. 3 for the circular arc, in Fig. 4 for the Von Mises airfoil and in Fig. 5 for the Kármán-Trefftz airfoil. The angle of attack is considered zero for the circular arc and 10° for the Von Mises and Kármán-Trefftz profiles.

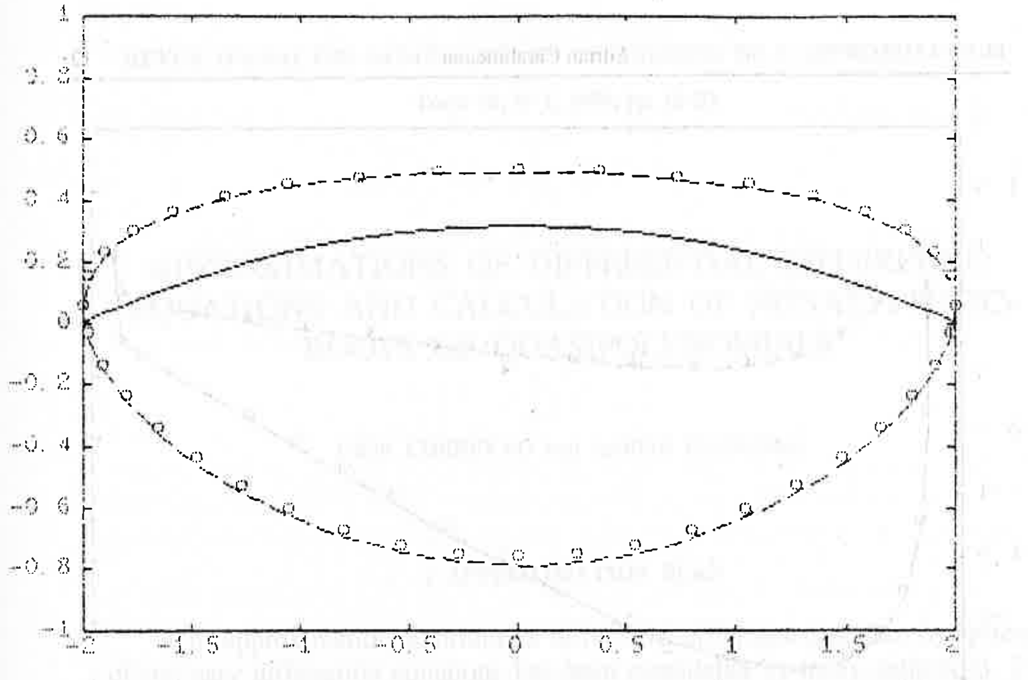

Fig. 3. – Circular arc at zero angle of attack.

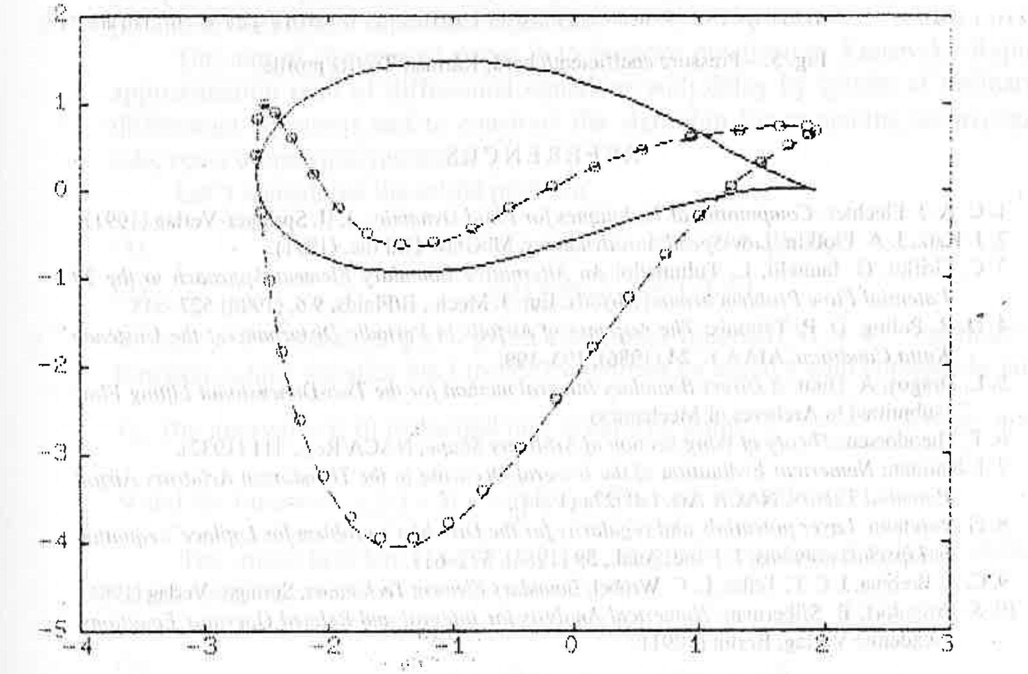

Fig. 4. – Von Mises profile.

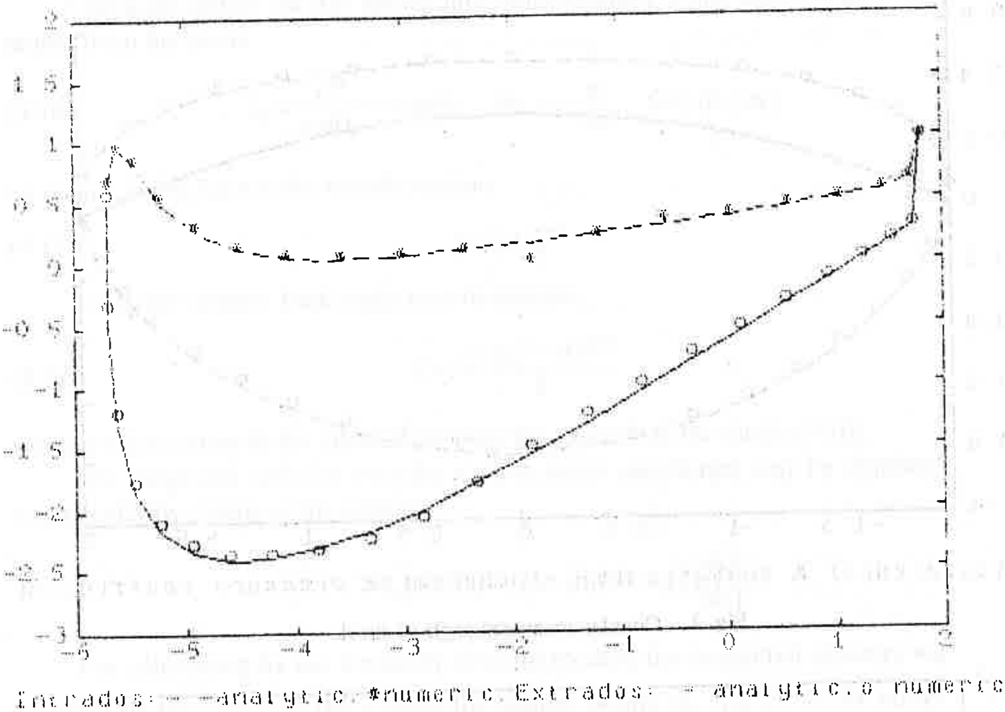

Fig. 5. – Pressure coefficient/chord; Kármán-Trefftz profile.

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