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APPROXIMATIONS OF DIFFERENTIAL DIFFERENCE EQUATIONS AND CALCULATION OF NONASYPTOTIC ROOTS OF QUASIPOLYNOMIALS

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1. APPROXIMATION PLAN

The approximation algorithm of differential difference equations by system of ordinary differential equations has been considered by many authors [1, 2] during the researches of control and stability problems in systems with delay. The most popular is Krasovsky N. N. and Yu. M. Repin [1] approximation system plan. This approximation system plan was applied to neutral type equations [3] and to differential functional equations.

The aim of the present paper is to improve precision of Krasovsky-Repin approximation plan of differential equations with delay by system of ordinary differential equations and to construct the algorithm for computing nonasymptotic roots of quasipolynomials.

Let's considered the initial problem

$$x'(t) = f(t, x(t), x(t-\tau)), \quad t \in [0, T], \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0], \tag{2}$$

where $\tau > 0$ – constant, $\varphi(t)$ – given continuous function; $f(t, u, v)$ – continuous function, which satisfies the Lipschitz condition by u and v with constant L_1 and L_2. The interval $[-\tau, 0]$ is divided into m parts by points $t_j = -\frac{j\tau}{m}$, $j = \overline{0, m}$, $m \in N$ and the functions $y_j(t) = x\left(t - \frac{\tau j}{m}\right)$, $j = \overline{0, m}$ are introduced.

The initial problem (1)–(2) in [1] is assigned to system of ordinary differential equations

$$\begin{aligned} z_0'(t) &= f(t, z_0(t), z_m(t)), \\ z_j'(t) &= \frac{m}{\tau}(z_{j-1}(t) - z_j(t)), \quad j = \overline{0, m}, \end{aligned} \tag{3}$$

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with initial conditions

(4) $$z_j(0)=\varphi\left(-\frac{j\tau}{m}\right), \quad j=\overline{0,m}.$$

N. N. Krasovsky shows [1] that uniformly for all bounded functions $\varphi(t)$

$$\max_{[0,T]}|x(t)-z_0(t)|=\alpha(m)\to 0, \quad \text{for } m\to\infty.$$

If the solution of the problem (1)–(2) satisfies the Lipschitz condition, then

(5) $$|\alpha(m)|\le\frac{K}{\sqrt{m}}, \quad K>0.$$

So, the replacing of delay equation (1) by system (3) is correct on fixed interval $[0, T]$ if m is taken large enough. We can consider system (3) as series of seccisive jointed delay elements [1]. Formally we can obtain it if we use two elements of the decomposition of function $y_{j-1}(t)=x\left(t-\frac{j\tau}{m}+\frac{\tau}{m}\right)$ in a Taylor series in $\left(t-\frac{\tau j}{m}\right)$ point environs. Now if we restrict to three elements of the decomposition in Taylor series

(6) $$y_{j-1}(t)=x\left(t-\frac{j\tau}{m}+\frac{\tau}{m}\right)=y_j(t)+\frac{\tau}{m}y_j'(t)+\frac{1}{2}\left(\frac{\tau}{m}\right)^2 y_j''(t)+\dots,$$

then we assigned equation (1) to a system of ordinary differential equations

(7) $$z_0'(t)=f(t,z_0,z_m),$$
$$\frac{1}{2}\left(\frac{\tau}{m}\right)^2 z_j''(t)+\frac{\tau}{m}z_j'+z_j=z_{j-1}, \quad j=\overline{1,m}.$$

Initial conditions for system (7) are

(8) $$z_0(0)=\varphi(0), \quad z_j(0)=\varphi\left(-\frac{j\tau}{m}\right), \quad z_j'(0)=\varphi'\left(-\frac{j\tau}{m}\right), \quad j=\overline{1,m}.$$

Further we show, that Caushy problem (7)–(8) approximates the problem (1)–(2) for delays equations and state the value of precision approximation.

2. DELAY ELEMENT APPROXIMATION

LEMMA 1. *Let's consider the system of linear differential equations*

(9) $$\frac{1}{2}\left(\frac{\tau}{m}\right)^2 z_1''+\frac{\tau}{m}z_1'+z_1=x(t)$$
$$\frac{1}{2}\left(\frac{\tau}{m}\right)^2 z_j''+\frac{\tau}{m}z_j'+z_j=z_{j-1}, \quad j=\overline{2,m}$$

with initial conditions

(10) $$z_j(0)=x\left(-\frac{j\tau}{m}\right), \quad z_j'(0)=x'\left(-\frac{j\tau}{m}\right), \quad j=\overline{1,m},$$

where $x(t)\in C^1[-\tau, T]$, $x'(t)$ satisfies the Lipschitz condition, τ, $T>0$ are constants. Then

(11) $$\left|z_j(t)-x\left(t-\frac{j\tau}{m}\right)\right|\le\frac{A}{m} \quad j=\overline{1,m}$$

is valid, where $A>0$ is constant which doesn't depends on j and m.

Proof. We assume that $x(t)\in C^2[-\tau, T]$ and consider the problem

(12) $$\frac{\tau^2}{2}z''+\tau z'+z=x(t), \quad z(0)=x(-\tau), \quad z'(0)=x'(-\tau).$$

Denote $y(t)=x(t-\tau)$ and estimate the value of difference $\varepsilon(t)=z(t)-y(t)$, which is the solution of the problem

(13) $$\varepsilon''(t)=\frac{2}{\tau}\varepsilon'(t)+\frac{2}{\tau^2}\varepsilon(t)=\varphi(t), \quad \varepsilon(t)=\varphi(t), \ \varepsilon(0)=0, \ \varepsilon'(0)=0,$$

where $\varphi(t)=\frac{2}{\tau^2}[x(t)-x(t-\tau)-x'(t-\tau)]-x''(t-\tau)$. If $x''(t)$ satisfies the Lipschitz conditions with constant K_2, then $|\varphi(t)|\le K_2\tau$. If $x'''(t)$ exists and is bounded by M_2, then $|\varphi(t)|\le\frac{1}{6}M_2\tau$.

For the solution of (13) we get

(14) $$\varepsilon(t)=\int_0^t K(t,s)\varphi(s)ds,$$

where $K(t,s)=\tau e^{\frac{-t-s}{\tau}}\sin\left(\frac{t-s}{\tau}\right)$.

Using the following property of function $\varphi(t)$ and $K(t, s)$ from (14) obtain

(15) $$|\varepsilon(t)|\le C\tau^3,$$

where $C=K_2$ or $C=M_2/6$.

Now we consider system of equations (6). Denote $y_j(t)=x\left(t-\frac{\tau j}{m}\right)$ and consider the difference $\varepsilon_j(t)=z_j(t)-y_j(t)$. For $\varepsilon_1(t)$ and according to (15) we get

$$|\varepsilon_1(t)|=|z_1(t)-y_1(t)|\le C\left(\frac{\tau}{m}\right)^3.$$

Continuing similarly, we receive

$$(16)\qquad |\varepsilon_j(t)| \le jC\left(\frac{\tau}{m}\right)^3 \le C\frac{\tau^3}{m^2}.$$

Now we do the conditions on $x(t)$ more weakly. We assume that $x'(t)$ satisfies the Lipschitz conditions with constant K_1 and $|x'(t)| < M_1$. Consider the smoothing function

$$x_1(t) = \frac{1}{h}\int_t^{t+h} x(s)\mathrm{d}s, \quad t \in [-\tau, T],$$

the second derivative of the function satisfies the Lipschitz condition with constant $\dfrac{2K_1}{h}$.

Let's estimate the value of function $x_2(t) = x(t) - x_1(t)$ and its derivative

$$(17)\qquad |x_2(t)| = \left|x(t) - \frac{1}{h}\int_t^{t+h} x(s)\mathrm{d}s\right| = \left|x(t) - \frac{1}{h}\int_t^{t+h}[x(t) - (s-z)x'(\theta s)]\mathrm{d}s\right| \le \frac{hM_1}{2}$$

$$(18)\qquad |x_2'(t)| = \left|x'(t) - \frac{1}{h}[x(t+h) - x(t)]\right| \le \frac{h}{2}K_1.$$

Consider problem (12), where $x(t) = x_1(t) + x_2(t)$. Let $z = z_1 + z_2$, where z_1 and z_2 are the solutions of problems

$$\frac{\tau^2}{2}z_1'' + \tau z_1' + z_1 = x_1(t), \quad z_1(0) = x_1(-\tau), \quad z_1'(0) = x_1'(-\tau),$$

$$\frac{\tau^3}{2}z_2'' + \tau z_2' + z_2 = x_2(t), \quad z_2(0) = x_2(-\tau), \quad z_2'(0) = x_2'(-\tau).$$

Estimate the difference $z(t) - x(t-\tau)$. We have $|z(t) - x(t-\tau)| \le$ $\le |z_1(t) + z_2(t) - x_1(t-\tau) - x_2(t-\tau)| \le |z_1(t) - x_1(t-\tau)| + |z_2(t)| + |x_2(t-\tau)|$. As function $x_1(t)$ is sufficiently smooth, the according to (15)

$$(19)\qquad |z_1(t) - x_1(t-\tau)| \le \frac{2K_1\tau^3}{h}.$$

For function $z_2(t)$, take into account (17), (18), we can receive estimation $|z_2(t)| \le hB$, where

$$(20)\qquad B = \frac{\tau K_1}{2} + M_1\left(1 + \frac{\tau^2}{2}\right).$$

For $x_2(t-\tau)$ the estimation (17) is true. Therefore $|z(t) - x(t-\tau)| \le \dfrac{2K_1\tau^3}{h} + 2Bh$. If we consider the system of equations (9), where $x(t) = x_1(t) + x_2(t)$, and estimate similarly, we receive $\left|z_j(t) - x\left(t - \dfrac{j\tau}{m}\right)\right| \le j\dfrac{2K_1\tau^3}{hm^3} + 2Bh \le \dfrac{2K_1\tau^3}{hm^2} + 2Bh$. Putting $h = \dfrac{\tau^{3/2}}{m}$, we'll have $\left|z_j(t) - x\left(t - \dfrac{j\tau}{m}\right)\right| \le \dfrac{2\tau^{3/2}(K_1 + B)}{m} \equiv \dfrac{A}{m}$. Lemma 1 is proved.

3. DELAY'S EQUATION APPROXIMATION

Consider the question about the closeness of the solutions of (1), (2) and (7), (8).

THEOREM 1. *Let's assume that the initial function* $\varphi(t) \in C^1[-\tau, 0]$ *and satisfies the "matching" condition*

$$(21)\qquad \lim_{s \to 0^-} \varphi'(s) = \varphi(0, \varphi(0), \varphi(-\tau))$$

function $f(t, u, v)$ *is continuous and satisfies the Lipschitz condition by* u *and* v *with constant* L_1, L_2. *Then it is true*

$$(22)\qquad \max_{s \in [0, T]} |x(s) - z_0(s)| = \alpha\left(\frac{1}{m}\right),$$

where $\lim_{r \to 0} \alpha(r) = 0$.

Proof. If the condition (21) is satisfied, then the solution of (1), (2), $x(t)$ is in $C^1[-\tau, T]$. Let's $z_j(t)$, $j = \overline{0, m}$ is the solution of differential equation system (7), (8). Let us denote $R_j(t) = \max\limits_{0 \le s \le t}\left|z_j(s) - x\left(s - \dfrac{j\tau}{m}\right)\right|$, $j = \overline{0, m}$.

Putting $z_j = z_j^{(1)} + z_j^{(2)}$, where $z_j^{(1)}$, $z_j^{(2)}$ are solutions of problems

$$(23)\qquad \begin{cases} \dfrac{\tau}{m}z_1'^{(1)} + z_1^{(1)} = x(t), \quad z_1^{(1)}(0) = z_1^0 = x\left(\dfrac{-\tau}{m}\right), \\ \dfrac{\tau}{m}z_2'^{(1)} + z_2^{(1)} = z_1^{(1)}, \quad z_2^{(1)}(0) = z_2^0 = x\left(\dfrac{-2\tau}{m}\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\tau}{m}z_m'^{(1)} + z_m^{(1)} = z_{m-1}^{(1)}, \quad z_m^{(1)}(0) = z_m^0 = x\left(\dfrac{-m\tau}{m}\right) \end{cases}$$

$$\left\{\begin{array}{l}\frac{\tau}{m}z_1'^{(2)}+z_1^{(2)}=z_0(t)-x(t), \quad z_1^{(2)}(0)=0,\\ \frac{\tau}{m}z_2'^{(2)}+z_2^{(2)}=z_2^{(2)}, \quad z_2^{(2)}(0)=0,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \frac{\tau}{m}z_m'^{(2)}+z_{m1}^{(2)}=z_{m-1}^{(2)}, \quad z_m^{(2)}(0)=0.\end{array}\right. \tag{24}$$

For system (23) the conditions of lemma 1 are fulfilled, then

$$\left|z_{j1}-x\left(t-\frac{j\tau}{m}\right)\right|\le\frac{A}{m}, \quad j=\overline{1,m}.$$

The solution of system (24) satisfies inequality $|z_{j2}(t)|\le R_0(t), \quad j=\overline{1,m}$. Therefore, we obtain

$$R_j(t)\le\frac{A}{m}+R_0(t), \quad j=\overline{1,m} \tag{25}$$

Let's perform (1) and (4) in integral form

$$x(t)=x(0)+\int_0^t f(s,x(s),x(s-\tau))ds$$

$$z_0(t)=x(0)+\int_0^t f(s,z_0(s),z_m(s))ds.$$

Using the property of function $f(t, u, v)$ and inequality (25) we obtain

$$|x(t)-z_0(t)|\le\int_0^t[L_1R_0(s)+L_2R_m(s)]ds\le\int_0^t\left[(L_1+L_2)R_0(s)+\frac{AL_2}{m}\right]ds.$$

Using Gronwall's Lemma [4] we obtain

$$R(t)=\max_{0\le s\le t}|x(s)-z_0(s)|\le\frac{AL_2}{(L_1+L_2)m}\left(e^{(L_1+L_2)t}-1\right).$$

From the last inequality it follows, that the solution $x(t)$ of the initial problem (1)–(2) is uniformly approximated by the function $z_0(t)$ that can be defined from approximate system (7)–(8) on any bounded interval $[0, T]$. Theorem 1 is proved.

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