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APPROXIMATION BY BOOLEAN SUMS OF POSITIVE LINEAR OPERATORS VI: MONOTONE APPROXIMATION AND GLOBAL SMOOTHNESS PRESERVATION

HEINZ H. GONSKA and JIA-DING CAO

0. INTRODUCTION

In a series of recent papers the present authors investigated the degree of approximation by Boolean sums of certain linear operators from various points of view (see, e.g., [5] and the references cited there). One of the subjects considered was the preservation of higher order convexity including monotonicity ([14]). In the present note we shall again deal with the preservation of monotonicity in combination with pointwise estimates for algebraic polynomial approximation, using a different point of view from that in [14]. Pioneering work in this direction was done by Lorentz and Zeller [26]; see also the excellent survey paper by Chalmers and Metcalf [15]. More recent contributions along these lines are papers by DeVore and Yu [18] and Leviatan [25].

Our approach will be simpler and more constructive than those in the latter papers, in the sense that we will *not use intermediate spline approximants*, but directly approximate the function $f \in C_{[-1,1]}$ using an algebraic convolution-type operator with appropriate properties. Furthermore, we will investigate the preservation of global smoothness of f as expressed by its modulus of continuity.

While several of the results below are quite general, we shall focus here on the investigation of the quantitative behavior of certain convolution-type operators W_{sn-s} (being special instances of more general convolution-type operators $G_{m(n)}$ and their Boolean sum modifications to be defined below. The mappings W_{sn-s} are defined constructively using appropriate trigonometric kernels D_{sn-s} which are obtained using a technique employed earlier by Beatson. For all our considerations, the concept of *bell-shapedness* of the kernels D_{sn-s} will be crucial. It will be shown that the original operators W_{sn-s} satisfy Timan-type inequalities, while for their modifications we have Telyakovskiĭ-, DeVore-Gopengauz-,

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and Dzjadyk-type inequalities. These names were derived from their historical context; it will be made clear below at each occurence what exactly is meant by them.

The following notation will be used in this paper. By $\mathbb{N} = \{1, 2, 3, \ldots\}$ we denote the set of natural numbers. For $n \in \mathbb{N}$, Π_n will be the set of algebraic polynomials of degree $\le n$. The symbol $C[-1, 1]$ will denote the space of real-valued continuos functions defined on the compact interval $[-1; 1]$; $C_{2\pi}$ will stand for its trigonometric counterpart. For a continuous function f, $\|f\|$ will always denote its sup norm. Furthermore, for $k \in \{1, 2\}$, $\omega_k(f, \cdot)$ will be the first and second order modulus of continuity, respectively. Throughout this paper, c, $\tilde{c}$ will mean absolute positive constants independent of f, x, and n. The constants c and $\tilde{c}$ may be different at different occurrences, even on the same line.

1. GENERAL CONCEPTS

The classical method of proving Jackson's theorem for $f \in C[-1, 1]$ uses convolution operators of the form

$$G(f, x) = (f * g)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos t) g[\cos(\theta - t)]\, dt,\ x = \cos\theta \in [-1, 1].$$

Here $g \in C[-1, 1]$ is a fixed function. Clearly, $G : C[-1, 1] \to C[-1, 1]$. Due to the fact that $f \circ \cos$, $g \circ \cos \in C_{2\pi}$ and that $g \circ \cos$ is even, one also has the representations

$$(1.1)\quad G(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos(\theta - t)) \cdot g(\cos t) dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos(\theta + t)) \cdot g(\cos t) dt.$$

If $g = g_m \in \Pi_m$ is given in its Čebyšev form

$$(1.2)\quad g_m(z) = \sum_{k=0}^{m} a_k \cdot T_k(z) \quad a_k \in \mathbb{R},\quad T_k(z) = \cos(k \cdot \arccos z),$$

then $G = G_m$ attains the form

$$G_m(f, x) = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(\cos t) \left\{ \sum_{k=0}^{m} a_k \cdot \cos(k(\theta - t)) \right\} dt =$$

$$= \sum_{k=0}^{m} a_k \left\{ \frac{2}{\pi} \cdot \int_0^{\pi} f(\cos t) \cdot \cos kt\, dt \right\} \cdot T_k(x).$$

Thus, $G_m : C[-1, 1] \to \Pi_m$. If $K_{m(n)}$ is an even trigonometric kernels of the form

$$K_{m(n)}(v) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cdot \cos kv,\ v \in \mathbb{R},$$

then

$$(1.3)\quad g_{m(n)}(z) := K_{m(n)}(\arccos z) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cdot T_k(z)$$

is of the form (1.2).We thus have the representations

$$G_{m(n)}(f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos t) K_{m(n)}[\arccos(\cos(\theta - t))] dt =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos t) K_{m(n)}(\arccos x - t) dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos(\arccos x + t)) K_{m(n)}(t) dt =$$

$$= \frac{1}{\pi}\int_0^{\pi} f(\cos t) dt + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cdot \left\{ \frac{2}{\pi} \cdot \int_0^{\pi} f(\cos t) \cdot \cos kt\, dt \right\} T_k(x).$$

If $K_{m(n)} \ge 0$, then $g_{m(n)}(z) \ge 0$, $z \in [-1, 1]$, so that $G_{m(n)}$ is a positive linear operator. In the sequel we shall exclusively discuss the case where $g_{m(n)}$ is given as in (1.3) with $K_{m(n)} \ge 0$.

Sometimes suitable modifications of the operators $G_{m(n)}$ where used in order to guarantee side conditions to be satisfied. To be more specific, we recall the definition of the *Boolean sum* of two linear operators P and Q, which is given by $P \oplus Q := P + Q - P \circ Q$ (subject to compatible domains and ranges of P and Q). Note that $\oplus$ is an associative operation, but is, in general, non-commutative. The use of Boolean sum modifications of operators $G_{m(n)}$ as introduced above is motivated by the following version of a theorem by Barnhill and Gregory (cf., e.g., [10, Theorem 2.1]).

THEOREM 1.1. *Let P and Q be linear operators mapping a function space G into a subspace H of G. Let G_0 be a subset of G, and let $\mathscr{L} = \{\ell\}$ be a set of linear functionals defined on H.*

(i) *Let $\ell(Pf) = \ell(f)$ for all $\ell \in \mathscr{L}$ and all $f \in H$.*
Then $\ell[(P \oplus Q)f] = \ell(f)$ for all $\ell \in \mathscr{L}$ and all $f \in H$.

(ii) *Let $Qf = f$ for all $f \in G_0$.*
Then $(P \oplus Q)f = f$ for all $f \in G_0$.

(iii) *Let f and Qf be in the set of all functions g such that $Pg = g$.*
Then $(P \oplus Q)f = f$.

(iv) *Let $(Id - Q)f$, $P \circ (Id - Q)f \in \ker \ell$, the kernel of ℓ. Then $\ell((P \oplus Q)f) = \ell(f)$.*

Proof. The proofs of (i) through (iii) were given in [10].

(iv) We have $\ell\,[(P\oplus Q)f] = \ell\,(Pf) + \ell\,(Qf) - \ell\,((P\circ Q)f) =$

$= \ell\,(P\circ(Id-Q)f) + \ell\,(Qf-f) + \ell\,(f) = 0 + 0 + \ell f = \ell\,(f).$ □

In the present note we will again consider special Boolean sums of approximation operators A (which we usually envision as being members of a particular sequence of such operators) with a fixed linear interpolation operator L.

For a compact interval $[a, b]$ and a function f defined on it, we denote by Lf the linear function interpolating f at a and b, i.e.,

$$L(f,x) := \frac{f(b)(x-a)+f(a)(b-x)}{b-a}, \quad a \le x \le b. \tag{1.4}$$

Let $A: C[a, b] \to C[a, b]$ be a linear operator. For $f \in C[a, b]$ and $a \le x \le b$, we introduce its modifications

$$\begin{aligned} A+(f,x) &:= (L\oplus A)(f,x) = (L + A - L\circ A)(f,x) = \\ &= A(f,x) + (b-a)^{-1}\{(x-a)\cdot[f(b)-A(f,b)] + (b-x)\cdot[f(a)-A(f,a)]\}, \end{aligned} \tag{1.5}$$

and

$$A^*(f,x) := (A\oplus L)(f,x) = A(f-Lf,x) + L(f,x). \tag{1.6}$$

As a consequence of Theorem 1.1, for the special situation at hand one has

COROLLARY 1.2. (Cao and Gonska [10, Corollary 2.2]) *The operator $A^+ = L\oplus A$ as given above has the following properties:*

(i) $A^+(f;c) = f(c)$ *for all* $f \in C[a, b]$ *and* $c = a$ *or* $c = b$;

(ii) $A(\Pi_1) \subset \Pi_1$, *then* $A^+ f = f$ *for all* $f \in \Pi_1$.

For the operators A^* introduced above, Theorem 1.1 gives

COROLLARY 1.3. *For the operator* $A^* = A\oplus L$, *we have the following:*

(i) $A(f-Lf;c) = 0$ *for* $c = a$ *or* $c = b$, *then* $A^*(f;c) = f(c)$.

(ii) $A^* f = f$ *for all* $f \in \Pi_1$.

Proof.

(i) Let $\ell = \varepsilon_c$ be the point evaluation functional at c, and $P := A$, $Q := L$. We verify that the sufficient conditions from Theorem 1.1 (iv) are satisfied. Note first that

$$\varepsilon_c((Id-L)f) = f(c) - L(f;c) = f(c) - f(c) = 0.$$

Secondly,

$$\varepsilon_c(A\circ(Id-L)f) = A((Id-L)f;c) = A(f-Lf;c) = 0,$$

so that $A^*(f;c) = f(c)$ immediately follows.

(ii) This is a direct consequence of Theorem 1.1 (ii).

Remark 1.4. The Boolean sum approach to imposing interpolatory side conditions at the end-points of $[a, b]$ has the disadvantage that the positivity of the operator A might be lost when passing to A^+ or A^*. For an example, see [13, Example 2.5]. A different and quite interesting approach to imposing such interpolations conditions without the loss of positivity was recently presented by I. Gavrea [19, 20]. It appears to be of interest to investigate a blend of Gavrea's technique with the Boolean sum approach.

2. PREVIOUS RESULTS

In the following lemmas we collect some of our earlier results concerning the quantitative behavior of the operators $G_{m(n)}$ and their Boolean sum modifications.

An important tool for proving Timan-type inequalities is

LEMMA 2.1. (Cao [7], [8, Theorem 1]) *Let $K_{m(n)}$ be a non-negative kernel as given above. Then for $-1 \le x \le 1$ and $f \in C[-1, 1]$,*

$$\left|f(x) - G_{m(n)}(f,x)\right| \le 2\omega_1\left[f, (1-\rho_{1,m(n)})|x| + \sqrt{2}\cdot\sqrt{1-\rho_{1,m(n)}}\cdot\sqrt{1-x^2}\right].$$

The following assertion gives some sufficient conditions under which polynomial Boolean sum operators A_n^* satisfy Telyakovskiĭ-type inequalities.

LEMMA 2.2. (Cao and Gonska [12, Theorem 3]) *Let $n \ge 1$ and $m(n) \in \mathbb{N}\cup\{0\}$ with $cn \le m(n) \le \tilde{c}n$, $n \ge 2$, for some constants c, $\tilde{c}$. Let $A_n: C[-1, 1] \to \Pi_{m(n)}$ be a sequence of linear algebraic polynomial operators. Suppose that for A_n we have the Timan-type estimate*

$$\left|A_n(f,x) - f(x)\right| \le c\omega_1\left(f, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \quad |x| \le 1.$$

Then for A_n^+, the Telyakovskiĭ-type estimate

$$\left|A_n^+(f,x) - f(x)\right| \le c\omega_1\left(f, \frac{\sqrt{1-x^2}}{n}\right), \quad |x| \le 1, \textit{ holds true}.$$

The assertion below states under which conditions one has inequalities of the *DeVore-Gopengauz-type*.

LEMMA 2.3. (Boss, Cao and Gonska [5, Corollary 1]) *Let $m(n) \ge 2$, $K_{m(n)}(v) \ge 0$, $0 < \varepsilon_n \le 1$, and let*

(i) $$1 - \rho_{1,m(n)} = O\left[\varepsilon_n^2\right],$$

(ii) $\frac{3}{2}-2\rho_{1,m(n)}+\frac{1}{2}\rho_{2,m(n)}=O\left[\varepsilon_n^4\right]$.

Then for $f\in C[-1,1]$,

$$\left|G^+_{m(n)}(f,x)-f(x)\right|\le C\cdot\omega_2\left(f;\varepsilon_n\cdot\sqrt{1-x^2}\right),\quad |x|\le 1.$$

Here the constant C is independent of f, x, and n.

The following lemma states the conditions under which one has *Dzjadyk-type inequalities*.

LEMMA 2.4. (Cao and Gonska [9]) *Let* $n\in\mathbb{N}$ *and* $K_{m(n)}(v)\ge 0$, $-1\le x\le 1$. *Then for* $f\in C[-1,1]$,

$$\left|f(x)-G^+_{m(n)}(f,x)\right|\le c\omega_2\left(f,\sqrt{1-\rho_{1,m(n)}}\right).$$

3. NOTES ON GENERALIZED BEATSON KERNELS

In this report [3], Beatson used Steklov means of order 1 to construct so-called bell-shaped trigonometric kernels based upon Jackson kernels of order $s\in\mathbb{N}$. To be more specific, recall

DEFINITION 3.1. (Lorentz and Zeller [26]) *A continuous function on* $[-\pi,\pi]$ *is called bell-shaped if it is even and if it decreases on* $[0,\pi]$.

The following property of bell-shaped functions will be useful below.

LEMMA 3.2. (Beatson [3, Lemma 2]) *Let the* 2π*-periodic function* g *be bell-shaped. Then for all* $t,x\in[0,\pi]$, *one has*

$$g(t-x)-g(t+x)\ge 0.$$

Beatson's construction to define bell-shaped kernels can be generalized as follows. With $K_{m(n)}$ given as above, we construct new trigonometric kernels as the first order Steklov means

$$\begin{aligned}D_{m(n)}(v)&:=\frac{n}{2\pi}\int_{v-\frac{\pi}{n}}^{v+\frac{\pi}{n}}K_{m(n)}(t)\mathrm{d}t=\\&=\frac{1}{2}+\sum_{k=1}^{m(n)}\frac{n}{2\pi}\rho_{k,m(n)}\int_{v-\frac{\pi}{n}}^{v+\frac{\pi}{n}}\cos kt\,\mathrm{d}t=\\&=\frac{1}{2}+\sum_{k=1}^{m(n)}\frac{n}{k\pi}\sin\frac{k\pi}{n}\rho_{k,m(n)}\cos kv=\\&=\frac{1}{2}+\sum_{k=1}^{m(n)}\lambda_{k,m(n)}\cos kv,\end{aligned}\tag{3.1}$$

where

$$\lambda_{k,m(n)}=\frac{n}{k\pi}\sin\frac{k\pi}{n}\rho_{k,m(n)},\quad 1\le k\le m(n).\tag{3.2}$$

LEMMA 3.3. *Let* $m(n)\in\mathbb{N}$. If, for $1\le k\le m(n)$, *we define* $\lambda_{k,m(n)}$ *as above, then the following hold:*

(i) If $1-\rho_{1,m(n)}=O\left(\frac{1}{n^2}\right)$, then $1-\lambda_{1,m(n)}=O\left(\frac{1}{n^2}\right)$,

(ii) If $1-\rho_{1,m(n)}=O\left(\frac{1}{n^2}\right)$, $1-\rho_{2,m(n)}=O\left(\frac{1}{n^2}\right)$, and

$$\frac{3}{2}-2\rho_{1,m(n)}+\frac{1}{2}\rho_{2,m(n)}=O\left(\frac{1}{n^4}\right),\text{ then }\frac{3}{2}-2\lambda_{1,m(n)}+\frac{1}{2}\lambda_{2,m(n)}=O\left(\frac{1}{n^4}\right).$$

Proof. We have $\lambda_{1,m(n)}=\frac{n}{\pi}\sin\frac{\pi}{n}\rho_{1,m(n)}$, i.e., $1-\lambda_{1,m(n)}=1-\frac{n}{\pi}\sin\frac{\pi}{n}\rho_{1,m(n)}=$ $=\left(1-\frac{n}{\pi}\sin\frac{\pi}{n}\right)+\frac{n}{\pi}\sin\frac{\pi}{n}(1-\rho_{1,m(n)})$.

By Taylor's formula $1-\frac{\sin t}{t}=O(t^2)$; furthermore, $0<\frac{\sin t}{t}\le 1$, $0<t\le\pi$.

Thus $1-\lambda_{1,m(n)}=O\left(\frac{1}{n^2}\right)$.

Also, from $\lambda_{2,m(n)}=\frac{n}{2\pi}\sin\frac{2\pi}{n}\rho_{2,m(n)}$, we have

$$\frac{3}{2}-2\lambda_{1,m(n)}+\frac{1}{2}\lambda_{2,m(n)}=\frac{3}{2}-\frac{2n}{\pi}\sin\frac{\pi}{n}\rho_{1,m(n)}+\frac{1}{2}\left(\frac{n}{2\pi}\sin\frac{2\pi}{n}\right)\rho_{2,m(n)}.\tag{3.3}$$

Since $\frac{\sin t}{t}=1-\frac{t^2}{6}+O(t^4)$, we obtain

$$\frac{n}{\pi}\sin\frac{\pi}{n}=1-\frac{\pi^2}{6n^2}+O\left(\frac{1}{n^4}\right),\text{ and}$$

$$\frac{n}{2\pi}\sin\frac{2\pi}{n}=1-\frac{4\pi^2}{6n^2}+O\left(\frac{1}{n^4}\right)=1-\frac{2\pi^2}{3n^2}+O\left(\frac{1}{n^4}\right).$$

From condition (ii), we find that

$$\rho_{1,m(n)}=O(1)\text{ and }\rho_{2,m(n)}=O(1);$$

from (3.3) and condition (ii), we have

$$\frac{3}{2}-2\lambda_{1,m(n)}+\frac{1}{2}\lambda_{2,m(n)}=$$

$$=\frac{3}{2}-2\left(1-\frac{\pi^2}{6n^2}+O\left(\frac{1}{n^4}\right)\right)\rho_{1,m(n)}+\frac{1}{2}\left(1-\frac{2\pi^2}{3n^2}+O\left(\frac{1}{n^4}\right)\right)\rho_{2,m(n)}=$$

$$=\frac{3}{2}-2\rho_{1,m(n)}+\frac{\pi^2}{3n^2}\rho_{1,m(n)}+O\left(\frac{1}{n^4}\right)+\frac{1}{2}\rho_{2,m(n)}-\frac{\pi^2}{3n^2}\rho_{2,m(n)}+O\left(\frac{1}{n^4}\right)=$$

$$=\left[\frac{3}{2}-2\rho_{1,m(n)}+\frac{1}{2}\rho_{2,m(n)}\right]+\frac{\pi^2}{3n^2}\left(\rho_{1,m(n)}-\rho_{2,m(n)}\right)+O\left(\frac{1}{n^4}\right)=$$

$$=O\left(\frac{1}{n^4}\right)+\frac{\pi^2}{3n^2}\left(\rho_{1,m(n)}-1+1-\rho_{2,m(n)}\right)+O\left(\frac{1}{n^4}\right)=O\left(\frac{1}{n^4}\right).\ \square$$

Matsuoka investigated the following Jackson kernels of higher orders (see [17, p. 79 ff.], [27]).

For $s\in\mathbb{N}$, let $K_{sn-s}(\nu):=c_{n,s}\left(\frac{\sin\left(\frac{n\nu}{2}\right)}{\sin\left(\frac{\nu}{2}\right)}\right)^{2s}$,

where $c_{n,s}$ is chosen so that $\pi^{-1}\int_{-\pi}^{\pi}K_{sn-s}(\nu)d\nu=1$. Thus,

$$K_{sn-s}(\nu)=\frac{1}{2}+\sum_{k=1}^{sn-s}\rho_{k,sn-s}\cos k\nu. \tag{3.4}$$

The kernels constructed in (3.1) and based upon K_{sn-s} will be denoted by D_{sn-s}.

In the next lemma it will be shown that the kernels D_{sn-s} are bell--shaped (as was already observed by Beatson without proof).

LEMMA 3.4. *Let* $n,\ s\in\mathbb{N}$. *Then for* $0<\nu<\pi$,

$$\frac{dD_{sn-s}(\nu)}{d\nu}\le 0.$$

Proof. If $0<\nu<\pi$, we have by definition

$$D_{sn-s}(\nu)=\frac{n}{2\pi}\int_{\nu-\frac{\pi}{n}}^{\nu+\frac{\pi}{n}}C_{n,s}\left(\frac{\sin\left(\frac{nt}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right)^{2s}dt.$$

Thus,

$$\frac{dD_{sn-s}(\nu)}{d\nu}=\frac{nC_{n,s}}{2\pi}\left\{\frac{\left[\sin\frac{n}{2}\left(\nu+\frac{\pi}{n}\right)\right]^{2s}}{\left[\sin\frac{1}{2}\left(\nu+\frac{\pi}{n}\right)\right]^{2s}}-\frac{\left[\sin\frac{n}{2}\left(\nu-\frac{\pi}{n}\right)\right]^{2s}}{\left[\sin\frac{1}{2}\left(\nu-\frac{\pi}{n}\right)\right]^{2s}}\right\}=$$

$$=\frac{nC_{n,s}}{2\pi}\left[\sin\frac{n}{2}\left(\nu-\frac{\pi}{n}\right)\right]^{2s}\times\left\{\frac{1}{\left[\sin\frac{1}{2}\left(\nu+\frac{\pi}{n}\right)\right]^{2s}}-\frac{1}{\left[\sin\frac{1}{2}\left(\nu-\frac{\pi}{n}\right)\right]^{2s}}\right\}=$$

$$=\frac{nK_{sn-s}\left(\nu-\frac{\pi}{n}\right)}{2\pi\left[\sin\frac{1}{2}\left(\nu+\frac{\pi}{n}\right)\right]^{2s}}\left\{\left[\sin\frac{1}{2}\left(\nu-\frac{\pi}{n}\right)\right]^{2s}-\left[\sin\frac{1}{2}\left(\nu+\frac{\pi}{n}\right)\right]^{2s}\right\}.$$

Since $a^{2s}-b^{2s}=(a^2)^s-(b^2)^s=(a^2-b^2)[(a^2)^{s-1}+(a^2)^{s-2}b^2+\ldots+(b^2)^{s-1}]$, the quantity $a^{2s}-b^{2s}$ agrees in sign with $a^2-b^2=(a-b)(a+b)$.

Hence $\left\{\left[\sin\frac{1}{2}\left(\nu-\frac{\pi}{n}\right)\right]^{2s}-\left[\sin\frac{1}{2}\left(\nu+\frac{\pi}{n}\right)\right]^{2s}\right\}$ has the same sign as

$$\left[\sin\frac{1}{2}\left(\nu-\frac{\pi}{n}\right)+\sin\frac{1}{2}\left(\nu+\frac{\pi}{n}\right)\right]\times\left[\sin\frac{1}{2}\left(\nu-\frac{\pi}{n}\right)-\sin\frac{1}{2}\left(\nu+\frac{\pi}{n}\right)\right]. \tag{3.5}$$

Lorentz and Zeller [26, p. 502] proved that if $0\le\alpha,\ \beta\le\frac{\pi}{2}$, then

$$\sin(\alpha+\beta)\ge|\sin(\alpha-\beta)|. \tag{3.6}$$

Thus

$$\frac{dD_{sn-s}(\nu)}{d\nu}\le 0,\ 0<\nu<\pi.\ \square$$

4. DEGREE OF MONOTONE APPROXIMATION

Let j be a natural number. The j-th forward difference of an $f\in C[-1,\ 1]$ with increment h is then given by

$$\Delta_h^j f(t):=\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}f(t+kh),\ 0\le h\le 2/j \text{ and } t\in[-1,\ 1-jh].$$

A function f is called *j-convex* if $f\in C[-1,\ 1]$ and all j-th forward differences $\Delta_h^j f(t)$, $0\le h\le 2/j$, are non-negative. Also, the function f is said to be *0-convex* if it is non-negative. Beatson [4, Theorem 1] proved the following lemma, which is essential for our purposes.

LEMMA 4.1. *Let $g(z) \in C[-1, 1]$ and j be a non-negative integer. The cone of j-convex functions is invariant under the operator $G(f) = f * g$ iff $g(z)$ is j-convex.*

We denote the operators from (1.1) based upon the trigonometric kernels $D_{sn-s}(v)$ by W_{sn-s}. Our next assertion is the theorem of Lorentz and Zeller [26] for the operators W_{sn-s}.

THEOREM 4.2. *Let $n \in \mathbb{N}$ and $s \geq 2$. Then for $f \in C[-1, 1]$ and $-1 \leq x \leq 1$,*

$$|f(x) - W_{sn-s}(f, x)| \leq c\omega_1\left(f, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right).$$

In addition, if f is 1-convex, then $W_{sn-s}(f, \cdot)$ is also 1-convex.

Proof. We take $h(z) := D_{sn-s}(\arccos z)$, $(z = \cos v, -1 \leq z \leq 1)$. Then $h(\cos v) = D_{sn-s}(v)$. By Lemma 3.4, we have

$$h'(\cos v)(-\sin v) = \frac{d}{dv} D_{sn-s}(v) \leq 0, \quad 0 < v < \pi;$$

hence $h'(\cos v) \geq 0$, i.e. $h'(z) \geq 0$, $-1 < z < 1$.

Thus $h(z)$ is an increasing function of z on $-1 \leq z \leq 1$. Using Lemma 4.1, it is clear that if $f(x)$ is a 1-convex function, then $W_{sn-s}(f, x)$ is also 1-convex. For the Matsuoka kernels $K_{sn-s}(v)$, we have ([17, p. 81]).

$$1 - \rho_{1,sn-s} = O\left(\frac{1}{n^2}\right), \quad s \geq 2.$$

From (3.1), (3.2) and Lemma 3.3 (i) for the associated kernels $D_{sn-s}(v)$, we find

$$1 - \lambda_{1,sn-s} = O\left(\frac{1}{n^2}\right), \quad s \geq 2.$$

Theorem 4.2 now follows from Lemma 2.1. □

The following two auxiliary results will be needed to show that the mappings $W^+_{sn-s} = L \oplus W_{sn-s}$ preserve monotonicity (while also satisfying a Telyakovskiĭ-type inequality).

LEMMA 4.3. (Cao and Gonska [14, Lemma 2.1]) *Let A be a positive operator mapping $C[a, b]$ into itself, and such that $A(1; x) = 1$. Let $f \in C[a, b]$ be increasing, and*

$$v(x) := \frac{1}{b-a}\{(x-a)[f(b) - A(f, b)] + (b-x)[f(a) - A(f, a)]\}.$$

Then v is also increasing on $[a, b]$.

LEMMA 4.4. *Let A be a positive linear operator mapping $C[a, b]$ into $C[a, b]$, with $A(1, x) = 1$. For $j \in \mathbb{N}$, let the cone of j-convex functions be invariant under the operator A. Then the cone of j-convex functions is also invariant under the operator A^+.*

Proof. If $j = 1$, and $f \in C[a, b]$ is increasing on $[a, b]$, by Lemma 4.3 it follows that the linear function $v(x)$ is increasing, i.e. $\Delta_h^1 v(x) \geq 0$, $0 \leq h \leq b-a$, $x \in [a, b-h]$. Under the conditions of Lemma 4.4, we have $\Delta_h^1 A(f, x) \geq 0$. Since $A^+(f, x) = A(f, x) + v(x)$, we have $\Delta_h^1 A^+(f, x) = \Delta_h^1 A(f, x) + \Delta_h^1 v(x) \geq 0$.

If $j \geq 2$, then $\Delta_h^j(\alpha x + \beta) = 0$, and thus

$$\Delta_h^j A^+(f, x) = \Delta_h^j A(f, x) + \Delta_h^j v(x) = \Delta_h^j A(f, x).$$

Hence, if $\Delta_h^j f(x) \geq 0$, from the assumption of Lemma 4.4 we have that

$$\Delta_h^j A(f, x) = \Delta_h^j A^+(f, x) \geq 0. \ \square$$

The *Telyakovskiĭ-type estimate* for the operators W^+_{sn-s} is next.

THEOREM 4.5. *Let $n \in \mathbb{N}$* and $s \geq 2$. *Then for $f \in C[-1, 1]$,*

$$|f(x) - W^+_{sn-s}(f, x)| \leq c\omega_1\left(f, \frac{\sqrt{1-x^2}}{n}\right), \quad |x| \leq 1;$$

in addition, if $f(x)$ is 1-convex, then so is $W^+_{sn-s}(f, x)$.

Proof. From Theorem 4.2, we have the Timan-type estimate

$$|f(x) - W_{sn-s}(f, x)| \leq c\omega_1\left(f, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \quad |x| \leq 1.$$

Using Lemma 2.2, it follows that

$$|f(x) - W^+_{sn-s}(f, x)| \leq c\omega_1\left(f, \frac{\sqrt{1-x^2}}{n}\right), \quad |x| \leq 1.$$

From Theorem 4.2, we see that if $f(x)$ is 1-onvex, then this is also true for $W_{sn-s}(f, x)$. Since $W_{sn-s}(1, x) = 1$ and W_{sn-s} are positive linear operators, by Lemma 4.4 we have that $W_{sn-s}(f, x)$ also is 1-convex. □

For operators W^+_{sn-s} based upon Jackson kernels of order $s \geq 3$, we also have the following *estimate of DeVore-Gopengauz-type* (see DeVore and Yu [18] for the first assertion of this type).

THEOREM 4.6. *Let* $n \geq 2$, $s \geq 3$, *and* $f \in C[-1, 1]$. *Then*

$$\left|f(x) - W^+_{sn-s}(f, x)\right| \leq c\omega_2\left(f, \frac{\sqrt{1-x^2}}{n}\right), \quad |x| \leq 1;$$

if f(x) is 1-convex, then so is $W^+_{sn-s}(f, x)$. □

Proof. From (3.4), we have that Matsuoka's kernel has the form

$$K_{sn-s}(v) = \frac{1}{2} + \sum_{k=1}^{sn-s} \rho_{k,sn-s} \cos kv.$$

Formulas (3.1) and (3.2) show that the kernel of the operator W_{sn-s} is given by

$$D_{sn-s}(v) = \frac{1}{2} + \sum_{k=1}^{sn-s} \lambda_{k,sn-s} \cos kv,$$

where

$$\lambda_{k,sn-s} = \frac{n}{k\pi} \sin \frac{k\pi}{n} \rho_{k,sn-s}, \quad 1 \leq k \leq sn-s.$$

If $s \geq 2$, then (see [17])

$$1 - \rho_{1,sn-s} = O\left(\frac{1}{n^2}\right).$$

From $K_{sn-s}(v) \geq 0$, we have (see [9])

$$0 < 1 - \rho_{2,sn-s} \leq 4\left(1 - \rho_{1,sn-s}\right) = O\left(\frac{1}{n^2}\right).$$

If $s \geq 3$, then (see [9])

$$\frac{3}{2} - 2\rho_{1,sn-s} + \frac{1}{2}\rho_{2,sn-s} = O(n^{-4}).$$

Using Lemma 3.3, we obtain

$$1 - \lambda_{1,sn-s} = O\left(\frac{1}{n^2}\right), \quad s \geq 2,$$

and

$$\frac{3}{2} - 2\lambda_{1,sn-s} + \frac{1}{2}\lambda_{2,sn-s} = O(n^{-4}), \quad s \geq 3.$$

Lemma 2.3 then implies

$$\left|f(x) - W^+_{sn-s}(f, x)\right| \leq c\omega_2\left(f, \frac{\sqrt{1-x^2}}{n}\right), \quad |x| \leq 1.$$

From Theorem 4.5, we know that if $f(x)$ is 1-convex, then $W^+_{sn-s}(f, x)$ is also 1-convex.

In the remainder of this section, we investigate the operators $W^*_{sn-s} = W_{sn-s} \oplus L$ and show in particular how these inherit shape-preservation and quantitative properties from the underlying operator W_{sn-s}. The next lemma deals with the preservation of 1-convexity by more general operators $A^* = A \oplus L$.

LEMMA 4.7. *Let A be a linear operator mapping* $C[a, b]$ *into* $C[a, b]$ *with* $A(1, x) = 1$, $x \in [a, b]$. *Suppose that* $h(x) := x - A(t, x)$ *is increasing on* $[a, b]$, *and let the cone of 1-convex functions be invariant under the operator A. Then the cone of 1-convex functions is invariant under the operator* $A^* = A \oplus L$.

Proof. Let $f(x) \in C[a, b]$ be increasing on $[a, b]$, and let L be given as above, i.e.,

$$L(f, x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

As noted above, we have

(4.1) $$A^*(f, x) = A(f - Lf, x) + L(f, x) = A(f, x) - A(Lf, x) + L(f, x).$$

Since $A(1, x) = 1$,

$$A(Lf, x) - L(f, x) =$$
$$= \frac{f(b) - f(a)}{b - a} A(t, x) + \left[-\frac{a(f(b) - f(a))}{b - a} + f(a)\right] A(1, x) - \frac{f(b) - f(a)}{b - a} x +$$
$$+ \frac{a(f(b) - f(a))}{b - a} - f(a) = \frac{f(b) - f(a)}{b - a}(A(t, x) - x).$$

Thus

(4.2) $$A^*(f, x) = A(f, x) + \frac{f(b) - f(a)}{b - a}(x - A(t, x)) = A(f, x) + \frac{f(b) - f(a)}{b - a} \cdot h(x).$$

Since $A(f, x)$ and $h(x)$ are both increasing on $[a, b]$, and $\frac{f(b) - f(a)}{b - a} \geq 0$, the function $A^*(f, x)$ also increases on this interval. □

LEMMA 4.8. *Let* $K_{m(n)}(v) \geq 0$, *and let the cone of 1-convex functions be invariant under the operator* $G_{m(n)}$. *Then the cone of 1-convex functions is invariant under the operator* $G^*_{m(n)}$.

Proof. We have (see [22]) the equalities

$$(4.3)\qquad G_{m(n)}(1,x)=1,\quad G_{m(n)}(t,x)=\rho_{1,m(n)}x,$$

and

$$1-\rho_{1,m(n)}=\frac{1}{\pi}\int_{-\pi}^{\pi}(1-\cos v)K_{m(n)}(v)dv>0.$$

Thus $1-\frac{d}{dx}G_{m(n)}(t,x)=1-\rho_{1,m(n)}>0$, so that $x-G_{m(n)}(t,x)$ increases. An application of Lemma 4.7 then implies Lemma 4.8. □

Our result on the degree of approximation by the monotonicity preserving mappings W^*_{sn-s} is the *Dzjadyk-type inequality* in

THEOREM 4.9. *For $n\in\mathbb{N}$ and $s\geq 2$, let W^*_{sn-s} be given as above. Then for $f\in C[-1,1]$ we have*

$$\left\|f-W^*_{sn-s}f\right\|\leq c\cdot\omega_2\left(f,\frac{1}{n}\right),$$

where $c=c(s)$ is independent of f and n. Furthermore, if f is 1-convex, then the same is true of $W^*_{sn-s}f$.

Proof. Note first that from Lemma 3.3 we have

$$1-\lambda_{1,sn-s}=O\left(n^{-2}\right),$$

provided this is true for $1-\rho_{1,sn-s}$. However, the latter fact was already used in the proof of Theorem 4.2, so that the inequality in terms of $\omega_2\left(f;\frac{1}{n}\right)$ immediately follows from Lemma 2.4. Furthermore, in Theorem 4.2 it was also shown that, for $s\geq 2$, the cone of 1-convex functions is invariant under W_{sn-s}, $n\in\mathbb{N}$. Lemma 4.8 then implies the full statement of Theorem 4.9.

Remark 4.10. The reader noted that the inequalities of Theorem 4.6 (for operators W^+_{sn-s}) and of Theorem 4.9 (for W^*_{sn-s}) are different, in the sense that the former is a pointwise estimate while the latter is uniform. Indeed, it is not possible to prove a Telyakovskiĭ-type inequality for the more general operators $G^*_{m(n)}=G_{m(n)}\oplus L$ where $G_{m(n)}$ is given in Section 1 and is based upon the positive kernel $K_{m(n)}$. To see this, consider the function $g(t)=1-t^2$, $t\in[-1,1]$. If we had a Telyakovskiĭ-type estimate as in Theorem 4.5, then this would mean that g is interpolated by $G^*_{m(n)}$ at $+1$, say. We have (see (4.2)) the representation

$$G^*_{m(n)}(g;1)=G_{m(n)}(g;1)+\frac{g(1)-g(-1)}{2}\cdot\left(1-G_{m(n)}(t;1)\right)=G_{m(n)}(g;1)=$$
$$=G_{m(n)}(1-t^2;1)=G_{m(n)}(1-(1-t^2)+1-2t;1)=$$
$$=1-\left(\frac{3}{2}-2\cdot\rho_{1,m(n)}+\frac{1}{2}\cdot\rho_{2,m(n)}\right)+1-2\cdot\rho_{1,m(n)}=\qquad\text{(cf. [9], [22])}$$
$$=\frac{1}{2}\left(1-\rho_{2,m(n)}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K_{m(n)}(t)dt-\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2t\cdot K_{m(n)}(t)dt=$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 2t)\cdot K_{m(n)}(t)dt>0.$$

Of course, this means that a DeVore-Gopengauz-type inequality also cannot hold. □

5. GLOBAL SMOOTHNESS PRESERVATION

Recently, the preservation of global smoothness of functions (as measured by a modulus of continuity or by *K*-functionals of various kinds) under certain linear operators was investigated quite intensively (see [16] for some recent results and numerous references). A central result concerning this question is the following

THEOREM 5.1. *(Anastassiou, Cottin and Gonska [1, Theorem 4]). Let $I=[a,b]$, $a<b$, be a compact interval, and $H: C(I)\to C(I)$, $H\neq 0$, be a linear operator satisfying the following conditions:*

(5.1) *The operator norm of H is bounded, i.e.,* $\|H\|<\infty$,

H maps $C^1(I)$ into $C^1(I)$, and

$$(5.2)\qquad \|(Hg)'\|\leq c\cdot\|g'\|\ \text{for all } g\in C^1(I).$$

Then for all $f\in C(I)$ and $t\geq 0$,

$$\omega_1(Hf;t)\leq\|H\|\cdot\tilde{\omega}_1\left(f;\frac{ct}{\|H\|}\right).$$

Here, $\tilde{\omega}_1$ is the least concave majorant of the modulus ω_1 with respect to the variable *t*.

Remark 5.2. Readers not familiar with the concept of the least concave majorant of a function *f* are referred to the monograph [24, p. 46 ff.] where this concept is discussed in detail. It will be crucial for all considerations which follow in this section.

We next investigate under which conditions operators $G_{m(n)}$ which are based upon kernels $K_{m(n)}$ preserve global smoothness of a function. For the moment, we assume only that we are dealing with an operator G of the form

(5.3) $$G(f;x)=\pi^{-1}\int_{-\pi}^{\pi} f(\cos s)\cdot K(\arccos x-s)\mathrm{d}s,$$

where the kernel K is in L_1 and is positive and even. Clearly, one can also write this as

$$G(f;x)=\pi^{-1}\int_{-\pi}^{\pi} f(\cos(\arccos x-t))\cdot K(t)\mathrm{d}t.$$

We first give a general estimate for $\left|\frac{\mathrm{d}}{\mathrm{d}x}G(f;x)\right|$. Writing $g:=f\circ\cos$, $\theta:=\arccos x$, G attains the form

$$G(f;x)=\pi^{-1}\int_{-\pi}^{\pi} g(\theta-t)\cdot K(t)\mathrm{d}t=:\bar{G}(g;\theta).$$

Note that $\bar{G}(g;\theta)$ is defined for all $g\in C_{2\pi}$ and $\theta\in\mathbb{R}$. From [6, Prop. 1.1.15], we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\bar{G}(g;\theta)=\pi^{-1}\cdot\int_{-\pi}^{\pi}\left\{\frac{\partial}{\partial\theta}g(\theta-t)\right\}\cdot K(t)\mathrm{d}t.$$

Here

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=-\frac{1}{\sqrt{1-x^2}},\text{ so that }\frac{1}{\frac{\mathrm{d}\theta}{\mathrm{d}x}}=-\sqrt{1-x^2}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\bar{G}(f\circ\cos,\arccos x)\cdot\sqrt{1-x^2}=-\frac{\mathrm{d}}{\mathrm{d}x}\bar{G}(f\circ\cos,\theta)=$$

$$=-\pi^{-1}\int_{-\pi}^{\pi}\frac{\partial}{\partial\theta}f(\cos(\theta-t))\cdot K(t)\mathrm{d}t=$$

$$=\pi^{-1}\int_{-\pi}^{\pi}\sin(\theta-t)\cdot f'(\cos(\theta-t))\cdot K(t)\mathrm{d}t=$$

$$=\pi^{-1}\int_{\theta+\pi}^{\theta-\pi}\sin s\cdot f'(\cos s)\cdot K(\theta-s)\mathrm{d}s=$$

$$=\pi^{-1}\int_{\theta-\pi}^{\theta+\pi}\sin s\cdot f'(\cos s)\cdot K(\theta-s)\mathrm{d}s=$$

$$=\pi^{-1}\int_{-\pi}^{\pi}\sin s\cdot f'(\cos s)\cdot K(\theta-s)\mathrm{d}s=$$

$$=\frac{1}{\pi}\cdot\left(\int_{-\pi}^{0}+\int_{0}^{\pi}\right)\sin s\cdot f'(\cos s)\cdot K(\theta-s)\mathrm{d}s=$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\sin s\cdot f'(\cos s)\cdot[K(\theta-s)-K(\theta+s)]\mathrm{d}s.$$

Thus,

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\bar{G}(f\circ\cos,\arccos x)\right|\cdot\sqrt{1-x^2}=$$

$$=\left|\frac{1}{\pi}\cdot\int_{0}^{\pi}\sin s\cdot f'(\cos s)\cdot[K(\theta-s)-K(\theta+s)]\mathrm{d}s\right|\le$$

$$\le\|f'\|\cdot\frac{1}{\pi}\cdot\int_{0}^{\pi}\sin s\cdot|K(\theta-s)-K(\theta+s)|\mathrm{d}s.$$

It thus remains to give a representation of

$$\pi^{-1}\int_{0}^{\pi}\sin s\cdot|K(\theta-s)-K(\theta+s)|\mathrm{d}s.$$

Assuming that K is bell-shaped, by Lemma 3.2 the latter quantity can be rewritten as

$$\pi^{-1}\int_{0}^{\pi}\sin s\cdot|K(\theta-s)-K(\theta+s)|\mathrm{d}s=$$

$$=\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi}\sin s\cdot[K(\theta-s)-K(\theta+s)]\mathrm{d}s=$$

$$=\frac{1}{2\pi}\cdot\left\{-\int_{+\pi}^{-\pi}\sin(\theta-\tilde{s})\cdot K(\tilde{s})\mathrm{d}\tilde{s}-\int_{+\pi}^{-\pi}\sin(\tilde{s}-\theta)\cdot K(\tilde{s})\mathrm{d}\tilde{s}\right\}=$$

$$=\frac{1}{2\pi}\cdot\left\{\int_{-\pi}^{\pi}[\sin(\theta-s)-\sin(s-\theta)]\cdot K(s)\mathrm{d}s\right\}=$$

$$=\frac{1}{\pi}\cdot\int_{-\pi}^{\pi}\sin(\theta-s)\cdot K(s)\mathrm{d}s=$$

$$=\frac{1}{\pi}\cdot\int_{-\pi}^{\pi}[\sin\theta\cdot\cos s-\cos\theta\cdot\sin s]\cdot K(s)\mathrm{d}s=$$

$$=\frac{1}{\pi}\left[\sin\theta\cdot\int_{-\pi}^{\pi}\cos s\cdot K(s)\mathrm{d}s-\cos\theta\cdot\int_{-\pi}^{\pi}\sin s\cdot K(s)\mathrm{d}s\right]=$$

$$=\frac{1}{\pi}\cdot\sin\theta\cdot\int_{-\pi}^{\pi}\cos s\cdot K(s)\mathrm{d}s=$$

$$=:\sin\theta\cdot\rho_1[K].$$

Note here that the inequality $0\le\pi^{-1}\int_{0}^{\pi}\sin s\cdot|K(\theta-s)-K(\theta+s)|\mathrm{d}s=\sin\theta\cdot\rho_1[K]$ implies $\rho_1[K]\ge 0$. Thus,

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}G(f;x)\right|\cdot\sqrt{1-x^2}=$$

$$=\left|\frac{\mathrm{d}}{\mathrm{d}x}\bar{G}(f\circ\cos,\arccos x)\right|\cdot\sqrt{1-x^2}\le\|f'\|\cdot\sin\theta\cdot\rho_1[K],$$

or

$$\left|\frac{d}{dx}G(f;x)\right| \le \rho_1[K]\cdot\|f'\|.$$

Recalling further that for operators G of the form (5.3) one has $\|G\| = \pi^{-1}\cdot\|K\|_{L_1[-\pi,\pi]}$, the above can be summarized as follows.

THEOREM 5.3. (cf. Anastassiou and Gonska [2, Theorem 4.1]) Let G be a convolution-type operator of the form (5.3) which is based upon the bell-shaped kernel $K \ne 0$. Then $\rho_1[K] \ge 0$, and for all $f \in C[-1, 1]$ and all $0 < \delta \le 2$, one has

$$\omega_1(Gf;\delta) \le \frac{1}{\pi}\|K\|_{L_1[-\pi,\pi]}\,\tilde{\omega}_1\left(f;\frac{\rho_1[K]\cdot\delta}{\pi^{-1}\cdot\|K\|_{L_1[-\pi,\pi]}}\right) \le$$

$$\le \left(\pi^{-1}\cdot\|K\|_{L_1[-\pi,\pi]} + \rho_1[K]\right)\cdot\omega_1(f;\delta). \quad \square$$

We now specialize K further by assuming that

$$K(t) = K_{m(n)}(t) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)}\cdot\cos kt$$

is a non-negative and bell-shaped trigonometric polynomial of degree $\le m(n)$. The operators G based upon these kernels will be denoted by $G_{m(n)}$. We thus have

$$\text{(5.4)} \qquad G_{m(n)}(f;x) = \pi^{-1}\int_{-\pi}^{\pi} f(\cos(\theta - t))\cdot K_{m(n)}(t)\,dt.$$

THEOREM 5.4. (cf. Anastassiou and Gonska [2, Theorem 4.2]) *Let $G_{m(n)}$ be a convolution-type operator as in (5.4). Then for all $f \in C[-1, 1]$ and all $0 \le \delta \le 2$, one has*

$$\omega_1(G_{m(n)}f;\delta) \le \tilde{\omega}_1(f;\rho_{1,m(n)}\cdot\delta) \le (1+\rho_{1,m(n)})\cdot\omega_1(f;\delta).$$

Proof. We note first that $\|K_{m(n)}\|_{L_1[-\pi,\pi]} = \pi$, so that $\|G_{m(n)}\| = 1$.

Furthermore,

$$\rho_1[K_{m(n)}] = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi}\cos s\cdot K_{m(n)}(s)\,ds =$$

$$= \frac{1}{\pi}\cdot\int_{-\pi}^{\pi}\cos s\cdot\left(\frac{1}{2} + \sum_{k=1}^{m(n)}\rho_{k,m(n)}\cos ks\right)ds =$$

$$= \frac{1}{\pi}\cdot\int_{-\pi}^{\pi}\cos^2 s\cdot\rho_{1,m(n)}(s)\,ds =$$

$$= \frac{1}{\pi}\cdot\rho_{1,m(n)}\cdot\left[\frac{1}{2}s + \frac{1}{4}\cdot\sin 2s\right]_{-\pi}^{\pi} = \rho_{1,m(n)}.$$

The inequality of Theorem 5.4 then follow directly from Theorem 5.3. $\square$

Remark 5.5.

By the example of the function e_1, it can be seen that the constant 1 figuring in front of $\tilde{\omega}_1(f;\rho_{1,m(n)}\cdot\delta)$ is best possible. $\square$

COROLLARY 5.6. (cf. Anastassiou and Gonska [2, Corollary 4.3]) *Under the above assumptions on $K_{m(n)}$, it can be easily verified that, in addition to $0 \le \rho_{1,m(n)}$, one also has $\rho_{1,m(n)} \le 1$. From this it follows that*

$$\omega_1(G_{m(n)}f;\delta) \le \tilde{\omega}_1(f;\delta) \le 2\cdot\omega_1(f;\delta). \quad \square$$

Remark 5.7. The left inequality of Corollary 5.6 shows that the Lipschitz classes $\mathrm{Lip}_A(\alpha; [-1, 1])$, $0 < \alpha \le 1$, are invariant under the operator $G_{m(n)}$. $\square$

For the kernels

$$D_{sn-s}(v) := \frac{n}{2\pi}\cdot\int_{-\pi/n}^{\pi/n} K_{sn-s}(v+t)\,dt = \frac{1}{2} + \sum_{k=1}^{sn-s}\lambda_{k,sn-s}\cdot\cos kv$$

it follows from Lemma 3.4 that these are bell-shaped. Recalling further that $\lambda_{1,sn-s} = \frac{n}{\pi}\cdot\sin\frac{\pi}{n}\cdot\rho_{1,sn-s}$, our conclusion for the operators G based upon Beatson's kernels D_{sn-s}, which we denote by W_{sn-s}, is as follows:

THEOREM 5.8. *Let W_{sn-s} be the convolution-type operator based upon D_{sn-s}, where D_{sn-s} denotes Beatson's modification of the Jackson kernel K_{sn-s}, $s \ge 1$. Then for all $f \in C[-1, 1]$ and all $0 \le \delta \le 2$,*

$$\omega_1(W_{sn-s}f;\delta) \le \tilde{\omega}_1(f;\lambda_{1,sn-s}\cdot\delta) \le \tilde{\omega}_1(f;\delta),$$

and also

$$\omega_1(W_{sn-s}f;\delta) \le \tilde{\omega}_1(f;\lambda_{1,sn-s}\cdot\delta) \le (1+\lambda_{1,sn-s})\cdot\omega_1(f;\delta) \le 2\cdot\omega_1(f;\delta).$$

Remark 5.9. Explicit representations for the convergence factors $\rho_{1,sn-s}$, $s \ge 1$, can be found in [21, p. 37f.] and in [27]. From these the corresponding $\lambda_{1,sn-s}$ of D_{sn-s} can be easily derived.

5.1. GLOBAL SMOOTHNESS PRESERVATION BY OPERATORS W^+_{sn-s}

In the present section we will show that the Boolean sum modifications W^+_{sn-s} of the operators $W_{sn-s} : C[-1, 1] \to \Pi_{sn-s}$ also preserve global smoothness in a certain sense. This fact is a consequence of the following more general statement concerning mappings of the type $A^+ = L \oplus A$, where L is given as in

(1.4) and A satisfies some mild additional assumptions. In the sequel, $e_i(x) := x^i$, $i \in \mathbb{N}_0$.

THEOREM 5.10. *Suppose that A is a positive linear operator mapping C[a, b] into itself, with* $Ae_0 = e_0$, $Ae_1 = \rho_1 \cdot e_1$. *Let L be given as above.*
Suppose, furthermore, that $A : C^1[a, b] \to C^1[a, b]$ *such that*

$$\|(Ag)'\| \le c \cdot \|g'\| \text{ for all } g \in C^1[a, b].$$

Then for all f in C[a, b] and all $t \ge 0$,

$$\omega_1(A^+ f; t) \le 3 \cdot \tilde{\omega}_1\left(f; \frac{[c+1-\rho_1]\cdot t}{3}\right) \le 3\left(1 + \frac{c+1-\rho_1}{3}\right) \cdot \omega_1(f; t).$$

For $c = \rho_1$, *this inequality reduces to*

$$\omega_1(A^+ f; t) \le 3 \cdot \tilde{\omega}_1\left(f; \frac{t}{3}\right) \le 4 \cdot \omega_1(f; t).$$

Proof. We verify the conditions of Theorem 5.1 for the operators $A^+ = L \oplus A$ subject to the additional assumptions expressed in Theorem 5.10. In order to verify the boundedness of A^+, note that $\|A^+\| = \|L + A - L \circ A\| \le \|L\| + \|A\| + \|L\| \cdot \|A\| = 3$. To verify (5.2) for A^+, note that one has, for $x \in [a, b]$, the representation

$$A^+(f; x) = A(f; x) + \frac{1}{b-a} \cdot \{(x-a)\cdot(f(b) - A(f; b)) + (b-x)\cdot(f(a) - A(f; a))\}.$$

Thus,

$$\frac{d}{dx} A^+(f, x) = \frac{d}{dx} A(f; x) + \frac{1}{b-a}\{[f(b) - A(f; b)] - [f(a) - A(f; a)]\}.$$

Hence,

$$\left|\frac{d}{dx} A^+(f, x)\right| \le \left|\frac{d}{dx} A(f; x)\right| + \frac{1}{b-a} \cdot \left\{\left|f(b) - A(f; b)\right| - \left|f(a) - A(f; a)\right|\right\}.$$

Since A is a positive linear operator with $Ae_0 = e_0$, the second term of the latter sum can be rewritten as

$$\frac{1}{b-a} \cdot \left\{\left|A(f(b); b) - A(f; b)\right| - \left|A(f(a); a) - A(f; a)\right|\right\} \le$$
$$\le \frac{1}{b-a}\left\{\|f'\| \cdot A\left(|e_1 - b|; b\right) + \|f'\| \cdot A\left(|e_1 - a|; a\right)\right\} =$$
$$= \frac{\|f'\|}{b-a}\{A(b - e_1; b) + A(e_1 - a; a)\}.$$

From the equalities $Ae_0 = e_0$ and $Ae_1 = \rho_1 \cdot e_1$, it follows that this is equal to

$$\frac{\|f'\|}{b-a} \cdot \{b - \rho_1 b + \rho_1 a - a\} = \|f'\| \cdot (1 - \rho_1).$$

From $\|(Ag)'\| \le c \cdot \|g'\|$, $g \in C^1[a, b]$, we finally have

$$\left|\frac{d}{dx} A^+(f, x)\right| \le c\|f'\| + (1-\rho_1)\|f'\| = (c + 1 - \rho_1) \cdot \|f'\|.$$

(Note at this point that $c + 1 - \rho_1 \ge 0$.)

An application of Theorem 5.1, with the constant c there replaced by $c + 1 - \rho_1$, gives

$$\omega_1(A^+ f; t) \le 3 \cdot \tilde{\omega}_1\left(f; \frac{(c+1-\rho_1)t}{3}\right),$$

an inequality then implying the remaining claims of Theorem 5.10. □

Remark 5.11.

(i) Since the operator A^+ reproduces linear functions, the example of the function e_1 shows that in the inequality $\omega_1(A^+ f; t) \le 3 \cdot \tilde{\omega}_1\left(f; \frac{t}{3}\right)$, equality occurs in a nontrivial case.

(ii) The inequality from (i) shows, furthermore, that the Lipschitz classes $\mathrm{Lip}_A(1; [-1, 1])$ are invariant under A^+. We do not know whether this is also true for the classes $\mathrm{Lip}_A(\alpha; [-1, 1])$, $0 < \alpha < 1$. □

For the operators W^+_{sn-s}, we get

COROLLARY 5.12. *Let* W_{sn-s}, $s \ge 1$, *be the positive linear operators introduced above. Then for all* $f \in C[-1, 1]$ *and all* $t \ge 0$, *one has*

$$\omega_1(W^+_{sn-s} f; t) \le 3 \cdot \tilde{\omega}_1\left(f; \frac{t}{3}\right) \le 4 \cdot \omega_1(f; t).$$

5.2. GLOBAL SMOOTHNESS PRESERVATION BY OPERATORS W^*_{sn-s}

Global smoothness is also preserved in a certain sense by operators A^* given by $A^* := A \oplus L$, where, more explicitly,

$$A^*(f, x) = A(f - Lf, x) + L(f, x), \tag{5.5}$$

with L again defined as above. For mappings of this type, we have the following general statement.

THEOREM 5.13. *Suppose that A is a bounded linear operator mapping C[a, b] into itself, with* $\|A\| = 1$, *and let L be given as above. Assume, furthermore, that*

(5.6) $$Ae_0 = e_0, \quad Ae_1 = \rho_1 \cdot e_1$$

and that $A : C^1[a, b] \to C^1[a, b]$ *such that*

$$\|(Ag)'\| \le c \cdot \|g'\| \text{ for all } g \in C^1[a, b].$$

Then for all f in C[a, b] and all $t \ge 0$,

$$\omega_1(A^* f; t) \le 3 \cdot \tilde{\omega}_1\left(f; \frac{(c + |1-\rho_1|)t}{3}\right) \le 3\left(1 + \frac{c + |1-\rho_1|}{3}\right) \cdot \omega_1(f; t).$$

Proof. It is again easy to show that the conditions of Theorem 5.1 are satisfied for A^* as defined above. In order to verify the boundedness of A^*, simply note that

(5.7) $$\|A^*\| = \|A - A \circ L + L\| \le \|A\| + \|A\| \cdot \|L\| + \|L\| = 3.$$

In order to verify (5.2) of Theorem 5.1, note first that A^* maps $C^1(I)$ into itself. Furthermore, from (5.5) we have for all f in $C[a, b]$ and all x in $[a, b]$ that

$$A^*(f, x) = A(f - Lf, x) + L(f, x) = A(f, x) - (A - Id)(Lf, x).$$

Thus, for $f \in C^1[a, b]$, we get

(5.8) $$\left|\frac{d}{dx} A^*(f, x)\right| \le \left|\frac{d}{dx} A(f, x)\right| + \left|\frac{d}{dx}(A - Id)(Lf, x)\right| \le \le c \cdot \|f'\| + \left|\frac{d}{dx}(A - Id)(Lf, x)\right|,$$

with c given as above. From the assumptions $Ae_0 = e_0$, $Ae_1 = \rho_1 \cdot e_1$, we arrive at

$$\begin{aligned}(A - Id)(Lf; x) &= \\ &= (A - Id)\left(\frac{1}{b-a} \cdot \{f(b)(t-a) + f(a)(b-t)\}; x\right) = \\ &= (A - Id)\left(\frac{1}{b-a} \cdot \{f(b) \cdot t - f(a) \cdot t\}; x\right) = \\ &= (A - Id)\left(\frac{f(b) - f(a)}{b-a} \cdot t; x\right) = \\ &= \frac{f(b) - f(a)}{b-a} \cdot (A - Id)(e_1; x) = \\ &= \frac{f(b) - f(a)}{b-a} (\rho_1 - 1)x \quad \text{according to (5.6).}\end{aligned}$$

Thus,

(5.9) $$\left|\frac{d}{dx}(A - Id)(Lf; x)\right| = \left|\frac{f(b) - f(a)}{b-a} \cdot (\rho_1 - 1)\right| \le \|f'\| \cdot |1 - \rho_1|.$$

Combining (5.8) and (5.9) we obtain

$$\left|\frac{d}{dx} A^*(f, x)\right| \le (c + |1 - \rho_1|) \cdot \|f'\|,$$

which then, in view of (5.7), gives the inequality of Theorem 5.13. □

If we choose $A = G_{m(n)}$ as given above, then the assumptions of Theorem 5.13 are satisfied with $\rho_1 = \rho_{1,m(n)}$. For the particular operators W^*_{sn-s} considered here, we have the following result concerning their preservation of global smoothness.

COROLLARY 5.14. *Let* W_{sn-s}, $s \ge 1$, *be given as above. Then for all* $f \in C[-1, 1]$ *and all* $t \ge 0$, *we have*

$$\omega_1(W^*_{sn-s} f; t) \le 3 \cdot \tilde{\omega}_1\left(f; \frac{t}{3}\right) \le 4 \cdot \omega_1(f; t).$$

Proof. As was mentioned earlier, the W_{sn-s} are positive linear operators satisfying $W_{sn-s}(e_0) = e_0$ and $W_{sn-s}(e_1) = \lambda_{1,sn-s} \cdot e_1$. It was also shown above that $\|(W_{sn-s} g)'\| \le \lambda_{1,sn-s} \cdot \|g'\|$ for all $g \in C^1[-1, 1]$. Recall (see Corollary 5.6) that $0 \le \lambda_{1,sn-s} \le 1$. These facts then give the inequalities of Corollary 5.14. □

Remark 5.15.

Since the operators W^*_{sn-s} reproduce linear functions (cf. Theorem 4.9), a statement analogous to that of Remark 5.11 *(i)* holds. Furthermore, the first inequality of Corollary 5.14 also expresses the fact that the classes $\text{Lip}_A(1; [-1, 1])$ are invariant under W^*_{sn-s}. We do not know if this is also the case for $\text{Lip}_A(\alpha; [-1, 1])$, $0 < \alpha < 1$. □

Open Problems

1. **Can the preservation of monotonicity be combined with that of positivity (while still having the Telyakovskiĭ-type estimate)? We refer to Problem # 1 in [23] in regard to this question. Gavrea has recently done some interesting work in this direction [20].**
2. Can the global smoothness preservation statements for A^+ (see Theorem 5.10) and A^* (c.f., Theorem 5.13) be improved with respect to the constants figuring there?
3. What can be said about global smoothness preservation by discretely defined operators as introduced in our earlier paper [11]?

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REFERENCES

[1] G. A. Anastassiou, C. Cottin, H. H. Gonska, *Global smoothness of approximating functions*. Analysis, **11** (1991), 43–57.

[2] G. A. Anastassiou, H. H. Gonska, *On stochastic global smoothness*. Submitted for publication.

[3] R. K. Beatson, *Bell-shape preserving convolution operators*. *Technical Report*, Dept. of Math., University of Otago, Dunedin, New Zealand, 1978.

[4] R. Beatson, *Joint approximation of a function and its derivatives*. In: Approximation Theory III (Proc. Int. Sympos. Austin 1980, ed. by E.W. Cheney), 199–206. New York, Academic Press, 1980.

[5] A. Boos, Jia-ding Cao, H. H. Gonska, *Approximation by Boolean sums of positive linear operators V: on the constants in DeVore-Gopengauz-type inequalities*. Submitted for publication. *Temporary reference:* Schriftenreihe des Instituts für Angewandte Informatik, **SIAI-EBS-4-93**, European Business School, 1993.

[6] P. L. Butzer, R. J. Nesse, *Fourier Analysis and Approximation (vol. I)*. Basel and Stuttgart, Birkhäuser, 1971.

[7] Jia-ding Cao, *Generalizations of Timan theorem, Lehnhoff theorem and Telyakovskiĭ theorem* (Chinese). Kexue Tongbao, **15** (1986), 1132–1135. Science Bulletin (English Edition), **32** (1987), 1225–1229.

[8] Jia-ding Cao, *Generalizations of Timan's theorem, Lehnhoff's theorem and Telyakovskiĭ's theorem*. Schriftenreihe des Fachbereichs Mathematik, **SM-DU-106**, Universität Duisburg, 1986.

[9] Jia-ding Cao, H. H. Gonska, *Approximation by Boolean sums of positive linear operators*. Rend. mat., **6** (1986), 525–546.

[10] Jia-ding Cao, H. H. Gonska, *Approximation by Boolean sums of positive linear operators II: Gopengauz-type estimates*. J. Approximation Theory, **57** (1989), 77–89.

[11] Jia-ding Cao, H. H. Gonska, *Approximation by Boolean sums of positive linear operators III: Estimates for some numerical approximation schemes*, Numer. Funct. Anal. Optim., **10** (1989), 643–672.

[12] Jia-ding Cao, H. H. Gonska, *Approximation by Boolean sums of linear operators: Telyakovskiĭ-type estimates*. Bull. Austral. Math. Soc., **42** (1990), 253–266.

[13] Jia-ding Cao, H. H. Gonska, *Approximation by Boolean sums of positive linear operators IV: shape preservation by DeVore-Gopengauz-type approximants. Schriftenreihe des Instituts für Angewandte Informatik*, **SIAI-EBS-2-92**, European Business School, 1992.

[14] Jia-ding Cao, H. H. Gonska, *Pointwise estimates for higher order convexity preserving polynomial approximation*. To appear in "J. Austral. Math. Soc. Ser. B". *Temporary references*: Schriftenreihe des Institutes für Angewandte Informatik, **SIAI-EBS-2-93**, European Business School, 1993.

[15] B. L. Chalmers, G. D. Taylor, *Uniform approximation with constraints*, Jahresber. Deutsch. Math.-Verein., **81** (1979), 49–86.

[16] C. Cottin, H. H. Gonska, *Simultaneous approximation and global smoothness preservation*. To appear in "Rend. Circ. Mat. Palermo". *Temporary reference:* Schriftenreihe des Instituts für Angewandte Informatik, **SIAI-EBS-1-93**, European Business School, 1993.

[17] R. A. DeVore, The Approximation of Continuous Functions by Positive Linear Operators. Berlin-Heidelberg-New York, Springer, 1972.

[18] R. A. DeVore, Xiang-ming Yu, *Pointwise estimates for monotone polynomial approximation*. *Constructive Approximation*, **1** (1985), 323–331.

[19] I. Gavrea, *On Gonska's problem concerning approximation by algebraic polynomials*. Manuscript, 1992.

[20] I. Gavrea, *Approximation by positive linear operators*. To appear in Revue Anal. Numér. Théor. Approx.

[21] E. Görlich, E. L. Stark, Über beste Konstanten und asymptotische Entwicklungen positiver Faltungsintegrale und deren Zusammenhang mit dem Saturationsproblem. Jahresber. Deutsch. Math.-Verein., **27** (1970), 18–61.

[22] H. H. Gonska, *Modified Pičugov-Lehnhoff operators*. In: Approximation Theory V (Proc. Int. Sympos. College Station 1986, ed. by C.K. Chui et al), New-York, Academic Press 1986, 355–358.

[23] H. H. Gonska, Xin-long Zhou, *Polynomial approximation with side conditions: recent results and open problems*. Submitted for publication. *Temporary reference:* Schriftenreihe des Instituts für Angewandte Informatik, **SIAI-EBS-7-93**, European Business School, 1993.

[24] S. G. Kreĭn, Ju. I. Petunin, E. M. Semenov, *Interpolation of Linear Operators*. *Providence*, RI: Amer. Math. Soc., 1982.

[25] D. Leviatan, *Pointwise estimates for convex polynomial approximation*, Proc. Amer. Math. Soc. **98** (1986), 471–474.

[26] G. G. Lorentz, K. Zeller, *Degree of approximation by monotone polynomials*, J. Approx. Theory, **1** (1968), 501–504.

[27] Y. Matsuoka, *On the approximation of functions by some singular integrals*, Tôhoku Math. J., **18** (1966), 13–43.

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European Business School
D – 65375 Oestrich-Winkel
Germany
Dept. of Mathematics, Fudan University
PRC – Shanghai 200433
People's Republic of China