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A NOTE ON HÖLDER'S TYPE INEQUALITIES AND CONCAVE FUNCTIONS

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Abstract. In this note there are some theorems and examples that emphasize the relation between Hölder's type inequalities and concave or convex function. In many cases, by the same conditions that are imposed on the sequences $(x_1, \dots, x_n)/(y_1, \dots, y_n)$ in order to get bounds for the Hölder's type sums $\sum_{i=1}^{n} x_i^{1/p} y_i^{1/q}$, $1/p + 1/q = 1$, $p > 1$, we get bounds for $\sum_{i=1}^{n} y_i f(x_i / y_i)$ when $f(x)$ is a concave function.

1. In [1] the following theorem was proved:

THEOREM A. *Let $f(x) \in C'$ be a concave function in $0 < a \le x$ and let the vectors $0 < \boldsymbol{x}$, $0 < \boldsymbol{y}$, $0 < \boldsymbol{d}$ satisfy*

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} d_i = 1. \tag{1}$$

Let

$$\begin{matrix} x_i / y_i \ge d_i / y_i \ge d_j / y_j \ge x_j / y_j \\ i = 1, \dots, m \qquad j = m+1, \dots, n. \end{matrix} \tag{2}$$

Then

$$\sum_{i=1}^{n} y_i f(x_i / y_i) \le \sum_{i=1}^{n} y_i f(d_i / y_i). \tag{3}$$

If we replace x_i by $w_i x_i$, y_i by $w_i y_i$ and d_i by $w_i d_i$, we get the following theorem:

THEOREM 1. *Let $f(x) \in C'$ be a concave function in $0 < a \le x$ and let the vectors $0 < \boldsymbol{x}$, $0 < \boldsymbol{y}$, $0 < \boldsymbol{d}$ and $0 < \boldsymbol{w}$ satisfy*

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(4) $$\sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} w_i d_i.$$

Let

(2) $$x_i / y_i \geq d_i / y_i \geq d_j / y_j \geq x_j / y_j \qquad i = 1, \ldots, m \quad j = m+1, \ldots, n.$$

Then

(5) $$\sum_{i=1}^{n} w_i y_i f(x_i / y_i) \leq \sum_{i=1}^{n} w_i y_i f(d_i / y_i).$$

We give here a new proof of Theorem 1 in which we shall use the following result:

THEOREM 2. *Let the vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *with elements from* (c, d), *and let the vector* $\boldsymbol{w} > 0$ *satisfy*

(6) $$\sum_{i=1}^{k} w_i a_i \geq \sum_{i=1}^{k} w_i b_i. \qquad k = 1, \ldots, n.$$

Let $f(x): (c, d) \to \mathbb{R}$ *be a concave sectionally smooth decreasing function if the vector* $\boldsymbol{b}$ *is decreasing then*

(7) $$\sum_{i=1}^{n} w_i f(a_i) \leq \sum_{i=1}^{n} w_i f(b_i).$$

Furthermore if in addition to (6)

(6a) $$\sum_{i=1}^{n} w_i a_i = \sum_{i=1}^{n} w_i b_i$$

holds, then (7) holds if $f(x)$ *is concave sectionally smooth function and if* $f(x)$ *is concave increasing function,* $\boldsymbol{b}$ *is increasing and instead of (6) we have*

$$\sum_{i=1}^{k} w_i a_i \leq \sum_{i=1}^{k} w_i b_i. \qquad k = 1, \ldots, n$$

then (7) holds too.

Proof of Theorem 2. For any values $u, v \in (c, d)$ because of the concavity of $f(x)$

$$f(u) - f(v) \geq f'_+(u)(u - v)$$

so we have

$$f(b_i) - f(a_i) \geq f'_+(b_i)(b_i - a_i)$$

or

$$\sum_{i=1}^{n} w_i f(b_i) - \sum_{i=1}^{n} w_i f(a_i) \geq \sum_{i=1}^{n} w_i (b_i - a_i) f'_+(b_i) =$$

$$= f'_+(b_n) \sum_{i=1}^{n} w_i (b_i - a_i) - \sum_{k=1}^{n-1} \sum_{i=1}^{k} w_i (b_i - a_i)(f'_+(b_{k+1}) - f'_+(b_k)) \geq 0.$$

The last inequality is a result of (6) and the fact that $f(x)$ is concave decreasing, or of (6) and (6a) and the concavity of $f(x)$.

Remark. In [6 Th3] G. H. Toader proved the following:
Let the vectors $0 \leq \boldsymbol{x}$, $0 \leq z$ be given, and let $0 < \boldsymbol{y}$ be increasing, then

$$\sum_{i=1}^{k} x_i^q z_i \leq \sum_{i=1}^{k} y_i^q z_i, \qquad k = 1, \ldots, n$$

implies

$$\sum_{i=1}^{k} x_i^p z_i \leq \sum_{i=1}^{k} y_i^p z_i, \qquad k = 1, \ldots, n$$

for $0 < p < q$.

It is easy to see that Theorem 2 is an extension of this theorem. We get Toader's theorem from Theorem 2 by the substitution $x_i^q = a_i$, $y_i^q = b_i$, $z_i = w_i$ where $f(x) = x^{p/q}$, $0 < p/q < 1$.

THEOREM 3. *Let* $f(x)$ *be a concave function for* $x > 0$ *and let the vectors* $0 < \boldsymbol{x}, 0 < \boldsymbol{d}, 0 < \boldsymbol{y}$ *and* $0 < \boldsymbol{w}$ *satisfy (4) and*

(8) $$\sum_{i=1}^{k} w_i x_i \geq \sum_{i=1}^{k} w_i d_i, \qquad k = 1, \ldots, n-1$$

if $(d_1/y_1, d_2/y_2, \ldots, d_n/y_n)$ *is decreasing, then (5) is valid.*

Proof. Set in Theorem 2 $w_i \to w_i y_i$, $a_i \to x_i / y_i$, $b_i \to d_i / y_i$, $i = 1, \ldots, n$.

Proof of Theorem 1. We consider a rearrangement of the vectors $\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{y}, \boldsymbol{w}$, denoted by $\boldsymbol{x}^*$, $\boldsymbol{d}^*$, $\boldsymbol{y}^*$ and $\boldsymbol{w}^*$, such that for each vector, a component (i) before the rearrangement, maintains its position, say component (j). The rearrangement is such that $\{d_i^* / y_i^*,\}$ is decreasing.
In fact with respect to (2) we shall have a rearrangement between the first m components of the vectors and between the rest of the n-m components. Also, such rearrangement keep condition (2), so we have

$$x_i^* / y_i^* \geq d_i^* / y_i^* \qquad i = 1, \ldots, m.$$

Multiplication by $w_i^* y_i^*$ and addition give

(9) $$\sum_{i=1}^{k} w_i^* x_i^* \geq \sum_{i=1}^{k} w_i^* d_i^*, \qquad k = 1, \ldots, m.$$

Also we have

$$d_j^* / y_j^* \geq x_j^* / y_j^* \qquad j = m+1, \ldots, n$$

therefore we get

(10) $$\sum_{j=k}^{n} w_j^* x_j^* \leq \sum_{j=k}^{n} w_j^* d_j^* \qquad j = m+1, \ldots, n.$$

Of course, we have

(11) $$\sum_{j=1}^{n} w_i^* x_i^* = \sum_{j=1}^{n} w_i^* d_i^*.$$

(10) and (11) give

$$\sum_{j=1}^{k} w_i^* x_i^* \geq \sum_{j=1}^{k} w_i^* d_i^*, \qquad k = m+1, \ldots, n-1.$$

So, x^*, d^*, y^* and w^* satisfy the conditions of Theorem 3, therefore we have

$$\sum_{j=1}^{n} w_i^* y_i^* f(x_i^* / y_i^*) \leq \sum_{j=1}^{n} w_i^* y_i^* f(d_i^* / y_i^*),$$

which is with respect to our kind of rearrangement the same as (5). This concludes the proof of Theorem 1.

It is easy to realize by both proofs of Theorem 1 and Theorem 2 that we get the following result:

THEOREM 4. *Let $f(x) \in C'$ be a concave decreasing function and let the vectors $0 < \mathbf{x}$, $0 < \mathbf{y}$, $0 < \mathbf{d}$ and $0 < \mathbf{w}$ satisfy:*

(12) $$\sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i d_i.$$

If (2) is valid, then

(13) $$\sum_{i=1}^{k} w_i y_i f(x_i / y_i) \leq \sum_{i=1}^{k} w_i y_i f(d_i / y_i), \qquad k = 1, \ldots, n.$$

COROLLARY 1. *Let $f(x)$ be a concave decreasing function and let $\mathbf{x} > 0$, $\mathbf{y} > 0$, $\mathbf{d} > 0$, $\mathbf{w} > 0$ satisfy (2) and (4). If $\mathbf{w}$ is decreasing, then*

$$\sum_{i=1}^{k} w_i y_i f(x_i / y_i) \leq \sum_{i=1}^{k} w_i y_i f(d_i / y_i), \qquad k = 1, \ldots, n.$$

Proof of Corollary 1. It is easy to verify that if $\mathbf{w} > 0$ is decreasing and

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} d_i, \qquad k = 1, \ldots, n.$$

Then

$$\sum_{i=1}^{k} w_i x_i \geq \sum_{i=1}^{k} w_i d_i, \qquad k = 1, \ldots, n$$

(see also [4], proof of 16.A.2a, page 405). Hence (12) is satisfied and the Corollary follows from Theorem 4.

DEFINITION. For an $X = (x_1, \ldots, x_n)$, $x_1, \ldots, x_n$ real numbers, let $x(1) \leq \ldots \leq x(n)$ denote the components of X in increasing order, and let $(x(1), \ldots, x(n))$ denote the decreasing rearrangement of $(x_1, \ldots, x_n)$.

COROLLARY 2. *Let $0 < f(x) \in C'$ be a concave decreasing function, and let $0 < g(y) \in C'$ be a decreasing function. Let*

$$(x(1), \ldots, x(n)), \quad (y(1), \ldots, y(n)), \quad (d(1), \ldots, d(n))$$

be the increasing rearrangement of $\mathbf{x} > 0$, $\mathbf{y} > 0$, $\mathbf{d} > 0$ respectively. Let

(1) $$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} d_i$$

and

(14) $$x(i)/y(i) \geq d(i)/y(i) \geq d(j)/y(j) \geq x(j)/y(j), \qquad i = 1, \ldots, m, \quad j = m+1, \ldots, n.$$

Then

(15) $$\sum_{i=1}^{n} y_i g(y_i) f(x_i / y_i) \leq \sum_{i=1}^{n} y(i) g(y(i)) f(x(i)/y(i)) \leq \sum_{i=1}^{n} y(i) g(y(i)) f(d(i)/y(i)),$$

and

$$\sum_{i=1}^{k} y(i) g(y(i)) f(x(i)/y(i)) \leq \sum_{i=1}^{k} y(i) g(y(i)) f(d(i)/y(i)), \qquad k = 1, \ldots, n.$$

Proof of Corollary 2. Let us define

(16) $$G(x, y) = y g(y) f(x / y).$$

It is easy to see that $\partial G/\partial x$ is increasing in y. Hence it is a L-superadditive function and therefore [4, p. 156] satisfies

$$\sum_{i=1}^{n} G(x_i, y_i) \le \sum_{i=1}^{n} G(x(i), y(i)) \tag{17}$$

(17) is the left side inequality in (15). The right side inequality follows from Corollary 1 as $(g(y(1)), \ldots, g(y(n)))$ is decreasing.

2. By very similar considerations as in the proof of theorem A in [1] we get the following additional results

THEOREM 5. *Let $0 < f(x) \in C'$ be a concave decreasing function and let $S(x) \in C''$ be a concave increasing function. Let the vectors $\boldsymbol{x} > 0$, $\boldsymbol{y} > 0$ and $\boldsymbol{d} > 0$, satisfy (1) and (14), then*

$$\sum_{i=1}^{n} S(y_i f(x_i / y_i)) \le \sum_{i=1}^{n} S(y(i) f(x(i)/y(i))) \le \sum_{i=1}^{n} S(y(i) f(d(i)/y(i))) \tag{18}$$

and

$$\sum_{i=1}^{k} S(y(i) f(x(i)/y(i))) \le \sum_{i=1}^{k} S(y(i) f(d(i)/y(i))), \quad k = 1, \ldots, n. \tag{19}$$

Examples. Let $\boldsymbol{y} > 0$ be a given vector satisfying

$$y(1) < d, \qquad \sum_{i=1}^{n} y_i = 1 \tag{20}$$

suppose that the set A of vectors $\boldsymbol{x} > 0$ satisfying

$$x(i) \ge d, \qquad \sum_{i=1}^{n} x_i = 1 \tag{21}$$

then, there is a vector $0 < \boldsymbol{d} = (d_1, \ldots, d_n)$ in A such that

$$d_i = d, \quad i = 1, \ldots, m, \tag{22}$$

$$d_i = k y(i), \quad i = m + 1, \ldots, n, \tag{23}$$

when m and k satisfy

$$\mathrm{d}/y(m) > (1 - md)/\left(1 - \sum_{i=1}^{m} y(i)\right) = k \ge \mathrm{d}/y(m+1) \tag{24}$$

and for this $\boldsymbol{d}$, Corollary 2 and Theorem 5 hold.

In [2, Theorem 8] it was proved that such $\boldsymbol{d}$ exists and it was proved that

$$\sum_{i=1}^{n} x(i)^{1/p} y(i)^{1/q} \le \sum_{i=1}^{n} d(i)^{1/p} y(i)^{1/q}, \quad q > 1, \quad 1/p + 1/q = 1.$$

This inequality may be considered as special case of corollary 2 for $g(y) = 1$, $f(x) = x^{1/p}$.

From Corollary 2 for $\boldsymbol{y}$, $\boldsymbol{x}$, $\boldsymbol{d}$, satisfying (20)–(24) we get the following inequality

$$\sum_{i=1}^{n} y_i \cos((\pi x_i)/(L y_i)) \le \sum_{i=1}^{n} y(i) \cos((\pi x(i))/(L y(i))) \le$$

$$\le \sum_{i=1}^{m} y(i) \cos((\pi d)/(L y(i))) + \left(\sum_{i=m+1}^{n} y(i)\right) \cos\left(\pi(1 - md)/L \left(\sum_{i=m+1}^{n} y_i\right)\right)$$

when L is large enough so that

$$0 \le (\pi x_i)/(L y_i), \quad (\pi x(i))/(L y(i)), \quad (\pi d(i))/(L y(i)) \le \pi/2,$$

$$i = 1, \ldots, n.$$

For M large enough so that

$$M - (x(i)/y(i))^2, \quad M - (x_i/y_i)^2, \quad M - (d_i/y(i))^2 \ge 0 \quad i = 1, \ldots, n, \quad r > 1,$$

we get

$$\sum_{i=1}^{n} y_i^{(1/r-1)} (M y_i^r - x_i^r)^{1/r} \le \sum_{i=1}^{n} y(i)^{(1/r-1)} (M y(i)^r - x(i)^r)^{1/r} \le$$

$$\le \sum_{i=1}^{m} y(i)^{(1/r-1)} (M y(i)^r - d^r)^{1/r} + \left(\sum_{i=m+1}^{n} y(i)\right)^{(1/r-1)} \left(M (y(i))^r - (1 - md)^r\right)^{1/r}.$$

3. As we saw in Chapters 1 and 2 the function $f(x) = x^{1/p}$, $p > 1$ for which $y_i f(x_i / y_i) = y_i^{1/q} x_i^{1/p}$, leads to Hölder's type inequalities may be extended in many cases by replacing it with a concave function. Here we show additional examples of results that may be extended to general concave (or convex function).

In [2, Theorem 5] it was proved that if $\boldsymbol{x}$ is an increasing vector and $\boldsymbol{y} \ge 0$ is a decreasing vector and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1, \tag{25}$$

then

$$\sum_{i=1}^{n} y_i^{1/q} x_i^{1/p} \le (1/n)^{1/q} \sum_{i=1}^{n} x_i^{1/p}. \tag{26}$$

Using the proof of Theorem A, (26) may be extended as follows

$$\sum_{i=1}^{n} y_i f(x_i / y_i) \le \sum_{i=1}^{n} (1/n) f(nx_i),$$

where f is a concave function and $\boldsymbol{x}$ is an increasing and $\boldsymbol{y} > 0$ is a decreasing vector satisfying (25).

In [3, Theorem 1] the following was proven:

Let $\boldsymbol{a} \ge 0$ and $\boldsymbol{b} \ge 0$ be given increasing vectors. Among all increasing vectors $\boldsymbol{d} > 0$ which satisfy

$$\sum_{i=1}^{k} d_i \le \sum_{i=1}^{k} b_i, \qquad k = 1, \dots, n,$$

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} b_i,$$

there exists a unique vector $\boldsymbol{t}$ for which the minimum of $\sum_{i=1}^{n} d_i^{-1}(a_i)^2$ is attained at $\boldsymbol{d} = \boldsymbol{t}$. This $\boldsymbol{t}$ satisfies

$$t_i / a_i \le t_{i+1} / a_{i+1}, \quad i = 1, \dots, n-1$$

and if

$$t_k / a_k \le t_{k+1} / a_{k+1}, \text{ then } \sum_{i=1}^{k} t_i = \sum_{i=1}^{k} b_i.$$

This theorem is extended if we recognize that

$$\sum_{i=1}^{n} d_i^{-1} a_i^2 = \sum_{i=1}^{n} d_i (a_i / d_i)^2 = \sum_{i=1}^{n} d_i f(a_i / d_i),$$

when we replace $f(x) = x^2$ by any convex function, the proof is the same as in [3, Theorem 1].

Here is another example for a possible extension from $f(x) = x^{1/p}$ to a general concave function. The following was proved in [5, Theorem 2]:

Suppose that $a, b, c, x_i, y_i \ge 0$, $i = 1, \dots, n$. Let p and q satisfy

$$1/p + 1/q = 1, \quad p > 1.$$

Then for

$$z_i = (a/b)^{q/p} y_i,$$

$$\frac{b + c\sum_{i=1}^{n} x_i^{1/p} y_i^{1/q}}{\left(a + c\sum_{i=1}^{n} x_i\right)^{1/p}} \le \frac{b + c\sum_{i=1}^{n} z_i^{1/p} y_i^{1/q}}{\left(a + c\sum_{i=1}^{n} z_i\right)^{1/p}}$$

equality holds if and only if $z_i = x_i$, $i = 1, \dots, n$.

By replacing $f(x) = x^{1/p}$ with a positive concave function we get the following generalization:

Suppose that $f(x) > 0$ and concave, a, b, c, x_i, $y_i > 0$, $i = 1, \dots, n$, $z_i = y_i / \left(g^{-1}(b/a)\right)$, where $g^{-1}(x)$ is the inverse function to $g(x) = xf(1/x)$, then

$$\frac{b + c\sum_{i=1}^{n} y_i f(x_i / y_i)}{f\left(\dfrac{a + c\sum_{i=1}^{n} x_i}{ag^{-1}(b/a) + c\sum_{i=1}^{n} y_i}\right)} \le \frac{b + c\sum_{i=1}^{n} y_i f(z_i / y_i)}{f\left(\dfrac{a + c\sum_{i=1}^{n} z_i}{ag^{-1}(b/a) + c\sum_{i=1}^{n} y_i}\right)}.$$

If $f(x)$ is strictly concave then equality holds if and only if $x_i = z_i$, $i = 1, \dots, n$.

We get this inequality by a straight forward use of the property of concave function. For instance, if $f(x) = x\ln(1/x)$, $0 < x < 1$, then $g(x) = \ln(x)$ and $g^{-1}(x) = e^x$ and so we get the following inequality:

$$\frac{b + c\sum_{i=1}^{n} x_i \ln(y_i / x_i)}{\left(a + c\sum_{i=1}^{n} x_i\right)\left[\ln\left(ae^{(b/a)} + c\sum_{i=1}^{n} y_i\right) - \ln\left(a + c\sum_{i=1}^{n} x_i\right)\right]} \le$$

$$\le \frac{b + c\sum_{i=1}^{n} z_i \ln(y_i / x_i)}{\left(a + c\sum_{i=1}^{n} z_i\right)\left[\ln\left(ae^{(b/a)} + c\sum_{i=1}^{n} y_i\right) - \ln\left(a + c\sum_{i=1}^{n} z_i\right)\right]}.$$

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