REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 28, N° 1, 1999, pp. 73–77

SET-VALUED SOLUTIONS FOR AN EQUATION OF JENSEN TYPE

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1. INTRODUCTION

Let X be a vector space. We denote by $\mathscr{P}_0(X)$ the collection of all nonempty subsets of X. For $A, B \in \mathscr{P}_0(X)$ and $\lambda \in \mathbb{R}$ we define the sets $A+B$ and λA by

$$(1)\qquad \begin{aligned} A+B &= \{x \mid x \in X,\ x = a+b,\ a \in A,\ b \in B\} \\ \lambda A &= \{x \mid x \in X,\ x = \lambda a,\ a \in A\}. \end{aligned}$$

The following properties ([4]) will be often used in the sequel. For every A, $B \in \mathscr{P}_0(X)$ and every $\lambda, \mu \in \mathbb{R}$ we have:

$$(2)\qquad \begin{aligned} \lambda(A+B) &= \lambda A + \lambda B \\ (\lambda+\mu)A &\subseteq \lambda A + \mu A. \end{aligned}$$

If A is a convex set and $\lambda\mu \geq 0$ then

$$(3)\qquad (\lambda+\mu)A = \lambda A + \mu A.$$

A set $K \subseteq X$ is said to be a convex cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for all $\lambda > 0$. If the zero vector from X, denoted by 0_X, belongs to K we say that K is a cone with zero in X.

Let Y be a topological vector space satisfying the T_0 separation axiom (in this paper we suppose that all topological vector spaces satisfy this axiom). We denote by $C(Y)$ and $CC(Y)$ the families of all compact, respectively of all compact convex sets of $\mathscr{P}_0(Y)$. For a set $A \subseteq Y$ the closure of A will be denoted by cl$A$.

Let p be a real number, $0 < p < 1$, X, Y be real vector spaces and K a convex cone in X. In this paper we are looking for solutions $F : K \to \mathscr{P}_0(Y)$ of the equation

$$(4)\qquad F((1-p)x + py) = (1-p)F(x) + pF(y).$$

1991 AMS Subject Clasification: 47H04.

For $p=\frac{1}{2}$ the equation (4) becomes Jensen equation. It is best known that real valued functions that satisfy Jensen equation are of the form $f = a + k$, where a is an additive function and k is a real number ([2]). Z. Fifer [1] prove that an analogous representation holds for set-valued functions when $K = [0, +\infty)$ and Y is a real Banach space. K. Nikodem ([3], [4]) give a characterization of the solutions of Jensen equation for set-valued functions with compact convex values in a real topological vector space. In this paper we prove that an analogous characterization holds for the equation (4).

2. CHARACTERIZATION FOR SET-VALUED SOLUTIONS OF EQUATION (4)

We start by proving an auxiliary lemma.

LEMMA 2.1. *Let X, Y be real vector spaces and K a convex cone with zero in X. If the set-valued function* $F : K \to \mathcal{P}_0(Y)$ *satisfies the equation (4) then*

$$F(x+y)+F(0_X)=F(x)+F(y) \tag{5}$$

for every $x, y \in K$.

Proof. For $x = y = 0_X$ in (4) we have

$$F(0_X)=(1-p)F(0_X)+pF(0_X), \tag{6}$$

and for $x = 0_X$, respectively $y = 0_X$ in (4) we have

$$\begin{aligned} F((1-p)x)&=(1-p)F(x)+pF(0_X), & x\in K\\ F(py)&=(1-p)F(0_X)+pF(y), & y\in K. \end{aligned} \tag{7}$$

Now let $u, v \in K$. We have from (4)

$$F(u+v)=F\left((1-p)\frac{u}{1-p}+p\frac{v}{p}\right)=(1-p)F\left(\frac{u}{1-p}\right)+pF\left(\frac{v}{p}\right) \tag{8}$$

and taking account of the relations (6) and (8) we have

$$F(u+v)+F(0_X)=(1-p)F\left(\frac{u}{1-p}\right)+pF(0_X)+pF\left(\frac{v}{p}\right)+(1-p)F(0_X)$$

and using the relations (7) it results

$$F(u+v)+F(0_X)=F(u)+F(v)$$

and the lemma is proved.

For the proofs of the theorems that follows we will use some results concerning the convergence of sequences of subsets of a topological vector space.

Let Y be a real topological vector space. We denote by $A_n \to A$ the convergence of a sequence of sets in $\mathcal{P}_0(Y)$ endowed with the Hausdorff topology.

LEMMA 2.2. ([4]) *Let Y be a real topological vector space,* $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 1}$ *two sequences from* $\mathcal{P}_0(Y)$ *and* $A \in \mathcal{P}_0(Y)$ *a bounded set.*

1. *If* $(A_n)_{n\geq 1}$ *is a decreasing sequence of closed sets and* $(B_n)_{n\geq 1}$ *is a decreasing sequence of compact sets, then*
$$\bigcap_{n\geq 1}(A_n+B_n)=\bigcap_{n\geq 1}A_n+\bigcap_{n\geq 1}B_n.$$
2. *If* $(A_n)_{n\geq 1}$ *is a decreasing sequence of compact sets, then* $A_n \to \bigcap_{n\geq 1} A_n$.
3. *If* $(A_n)_{n\geq 1}$ *is an increasing sequence of subsets of a compact set, then* $A_n \to \mathrm{cl}\bigcup_{n\geq 1} A_n$.
4. *If* $A_n \to A$ *and* $B_n \to B$, *then* $A_n + B_n \to A + B$.
5. *If* $A_n \to A$ *and* $B_n \to B$, *then* $\mathrm{cl}A = \mathrm{cl}B$.
6. *The set-valued function* $G : \mathbb{R} \to \mathcal{P}_0(Y)$, $G(t) = tA$, $t \in \mathbb{R}$, *is continuous on* $\mathbb{R}$.

THEOREM 2.1. *Let X be a real vector space, Y a real topological vector space and K a cone with zero in X. If an set-valued function* $F : K \to CC(Y)$ *satisfies the equation (4) then there exists an additive set-valued function* $A : K \to CC(Y)$ *and a set* $B \in CC(Y)$ *such that* $F(x) = A(x) + B$ *for every* $x \in K$.

Proof. Assume that F satisfies the equation (4) and let $\alpha \in F(0_X)$. Then the set-valued function $G : K \to CC(Y)$, $G(x) = F(x) - \alpha$, $x \in K$, satisfies the equation (4) and $0_Y \in G(0_X)$. Then in view of Lemma 2.1

$$G(x+y)+G(0_X)=G(x)+G(y) \tag{9}$$

for all $x, y \in K$.

It can be easily proved by induction that we have

$$G(nx)+(n-1)G(0_X)=nG(x)$$

for all $x \in K$ and all $n \in \mathbb{N}$.

From (9) it results that

$$\frac{1}{2^n}G(2^n x)+\left(1-\frac{1}{2^n}\right)G(0_X)=G(x) \tag{10}$$

for all $n \geq 0$ and all $x \in K$.

Now let $x \in K$ fixed. The sequence $(G_n(x))_{n\geq 0}$ given by the relation

$$G_n(x) = \frac{1}{2^n} G(2^n x), \quad n \geq 0,$$

is decreasing. Indeed, taking account that $0_Y \in G(0_X)$ it results that

$$G_{n+1}(x) = \frac{1}{2^{n+1}} G(2^{n+1} x) \subseteq \frac{1}{2^{n+1}} \left(G(2^{n+1} x) + G(0_X) \right) =$$
$$= \frac{1}{2^{n+1}} \left(G(2 \cdot 2^n x) + G(0_X) \right) = \frac{1}{2^{n+1}} \cdot 2G(2^n x) = \frac{1}{2^n} G(2^n x) = G_n(x)$$

for all $n \geq 0$.

Let $A(x) = \bigcap_{n\geq 0} G_n(x)$. Since G has compact and convex values it results that $A(x) \in CC(Y)$. We prove that the set-valued function $A : K \to CC(Y)$ is additive.

Let $x, y \in K$. We have:

$$G_n(x+y) + \frac{1}{2^n} G(0_X) = \frac{1}{2^n} G\left(2^n (x+y)\right) + \frac{1}{2^n} G(0_X) =$$
$$= \frac{1}{2^n} \left(G(2^n x + 2^n y) + G(0_X) \right) = \frac{1}{2^n} \left(G(2^n x) + G(2^n y) \right) = \tag{11}$$
$$= \frac{1}{2^n} G(2^n x) + \frac{1}{2^n} G(2^n y) = G_n(x) + G_n(y).$$

In view of Lemma 2.2 we have

$$\frac{1}{2^n} G(0_X) \to \{0_Y\}$$
$$G_n(x) \to A(x)$$
$$G_n(y) \to A(y)$$
$$G_n(x+y) \to A(x+y).$$

By Lemma 2.2 and the relation (10) it follows that

$$\text{cl}\left(A(x+y) + \{0_Y\} \right) = \text{cl}\left[A(x) + A(y) \right]$$

and taking account that A has compact values it results that

$$A(x+y) = A(x) + A(y),$$

hence A is an additive set-valued function.

For the sequence $\left(\left(1 - \frac{1}{2^n} \right) G(0_X) \right)_{n\geq 0}$ we have in view of Lemma 2.2

$$\left(1 - \frac{1}{2^n} \right) G(0_X) \to G(0_X).$$

Now taking the limit in (10) we have

$$\text{cl}\left(A(x) + G(0_X) \right) = \text{cl}\, G(x)$$

and since the values of G and A are compact it follows that $G(x) = A(x) + G(0_X)$, $x \in K$, and denoting $B = G(0_X) + \alpha \in CC(Y)$ we obtain

$$F(x) = A(x) + B, \ x \in K.$$

The converse of Theorem 2.1 holds for $p \in Q$.

Remark. Let X, Y be real vector spaces, $A : X \to \mathcal{P}_0(Y)$ an additive set-valued function with convex values and $B \subseteq Y$ a convex set. Then the set-valued function $F : X \to \mathcal{P}_0(Y)$, $F(x) = A(x) + B$ for all $x \in X$, satisfies the equation (4) with $p \in Q$.

Proof. We have immediately $A(qx) = qA(x)$, for all $x \in X$, $q \in Q$, $q > 0$, and

$$F\left((1-p)x + py \right) = A\left((1-p)x + py \right) + B = A\left((1-p)x \right) + A(py) + B =$$
$$= (1-p)A(x) + pA(y) + (1-p)B + pB = (1-p)F(x) + pF(y)$$

for all $x, y \in K$.

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Received February 25, 1998

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