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# ON SOME REMARKABLE POSITIVE POLYNOMIAL OPERATORS OF APPROXIMATION

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***Dedicated to Professor H. H. Gonska on the occasion of his 50$^{th}$ birthday***

**Abstract.** In this paper there are investigated new approximation properties of a Bernstein type operator, depending on two real parameters $c$ and $d$ $(0 \le c \le d)$, introduced in 1969 by the first author [19].

A basic result consists in finding the maximum value (2.3) of the mean square error (2.1)–(2.2) of this operator. By using it, is constructed a best quadrature formula, which can be obtained also by means of the polynomial $S_m f$, defined at (3.5).

In the last part of the paper there are established quantitative estimations of approximation in terms of first and second order moduli of smoothness.

## 1. INTRODUCTION

**1.1.** It is known that polynomial approximation represents one of the most beautiful and important part of the constructive theory of functions.

The Lagrange interpolating polynomials have a great practical interest in numerical analysis and approximation theory. Unfortunately they do not always provide uniform convergent sequences of approximation for any continuous function on a compact interval $[a, b]$ of the real axis, no matter how the nodes are prescribed (see, e.g., E. W. Cheney [11]).

In 1905 E. Borel [9] proposed that for the approximation of a function $f \in C[0,1]$ to construct a polynomial having an expression similar with the Lagrange interpolating polynomial, corresponding to $m + 1$ nodes from $[0, 1]$. Namely, in the case of the equally spaced nodes, it has to be of the form

$$(Q_m f)(x) = \sum_{k=0}^{m} q_{m,k}(x)\, f\left(\frac{k}{m}\right),$$

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where $q_{m,k}$ are appropriate polynomials of degree $m$, which permit that $Q_m f$ to achieve a prescribed accuracy in the process of approximation of the function $f$.

Following the Borel ideas, S. N. Bernstein [7] has the merit to select, in 1912, for $q_{m,k}$ the basic polynomials

$$p_{m,k}(x)=\binom{m}{k}x^k(1-x)^{m-k}. \tag{1.1}$$

It should be mentioned that he was inspired by the binomial probability distribution and that he has investigated the convergence of the polynomials

$$(B_m f)(x)=B_m(f(t);x)=\sum_{k=0}^{m}p_{m,k}(x)f\left(\frac{k}{m}\right) \tag{1.2}$$

by using the weak Bernoulli law of large numbers.

**1.2.** In 1969 the first author has introduced in [19] the following generalization of the Bernstein polynomials

$$P_m^{(c,d)}(f(t);x)=\left(P_m^{(c,d)}f\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)f\left(\frac{k+c}{m+d}\right), \tag{1.3}$$

where $c$ and $d$ are real parameters, independently of $m$, satisfying the relations: $0\le c\le d$.

This polynomial is characterized by the fact that it uses equally spaced nodes, with the step $h=\frac{1}{m+d}$ and the starting point $x_0=\frac{c}{m+d}$. If $0<c\ne d$ then it does not coincide at any node with the function $f$; if $c=0$ and $d\ne c$ then it coincides with $f$ at $x_0=0$, while if $0<c=d$ then it coincides with $f$ at $x_m=1$. For $c=d=0$ we obtain the classical Bernstein polynomial $B_m f$, which coincides with $f$ at $x_0=0$ and $x_m=1$.

In the monograph of F. Altomare and M. Campiti [3] the operator defined at (1.3) was called: „the operator of Bernstein-Stancu" (pag. 117 and 220).

In the paper [19] of the author there was established the following representation of (1.3) in terms of finite differences

$$\left(P_m^{(c,d)}f\right)(x)=\sum_{j=0}^{m}\binom{m}{j}\left(\Delta^j_{\frac{1}{m+d}}f\right)\left(\frac{c}{m+d}\right)x^j, \tag{1.4}$$

as well as an expression by means of divided differences

$$\left(P_m^{(c,d)}f\right)(x)=\sum_{j=0}^{m}m^{[j]}\left[\frac{c}{m+d},\frac{c+1}{m+d},\dots,\frac{c+j}{m+d};f\right]\left(\frac{x}{m+d}\right)^j,$$



where $m^{[j]}=m(m-1)\dots(m-j+1)$, the brackets representing the symbol of divided differences.

**1.3.** For the monomials $e_0, e_1$ and $e_2$, where $e_j(t)=t^j$ $(j\ge 0)$, where $t\in[0,1]$, we have

$$P_m^{(c,d)}e_0=e_0\,;\quad \left(P_m^{(c,d)}e_1\right)(x)=x+\frac{c-dx}{m+d},$$
$$\left(P_m^{(c,d)}e_2\right)(x)=x^2+\frac{1}{(m+d)^2}[mx(1-x)+(c-dx)(2mx+dx+c)]. \tag{1.5}$$

Because for these „test functions" we have

$$\lim_{m\to\infty}P_m^{(c,d)}e_j=e_j\quad (j=0,1,2),$$

uniformly on the interval [0, 1], according to the Bohman-Korovkin convergence criterion there was possible to state the following result: *if* $f\in C[0,1]$ *then the sequence of polynomials* $\left(P_m^{(c,d)}f\right)$ *converges uniformly to the function f on the interval* [0, 1].

**1.4.** We mention also that for the eigenvalues of the operator $P_m^{(c,d)}$ we have obtained the following expressions

$$\lambda_{m,r}(P_m^{(c,d)})=\frac{m^{[r]}}{(m+d)^r}=\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{r-1}{m}\right)\left(\frac{m}{m+d}\right)^r,$$

where $r=0,1,\dots,m$. One can see that these quantities do not depend on the value of the parameter $c$.

In the case of Bernstein operator ($c=d=0$) the point spectrum was first found in [10]. In [6] it has been given a characterization of $B_m$ by using the eigenvalue $\lambda_{m,2}=\lambda_{m,2}(B_m)$.

Because in our case

$$\lambda_{m,2}\left(P_m^{(c,d)}\right)=\left(1-\frac{1}{m}\right)\left(1+\frac{d}{m}\right)^{-2}\le\lambda_{m,2}(B_m)=1-\frac{1}{m},$$

by using a theorem given in [6], we conclude that the best result can be achieved when $c=d=0$, that is in the case of Bernstein operator $B_m$.

**1.5.** The Bleimann, Butzer, Hahn (BBH) rational operator [8], given by

$$(L_m f)(x)=\sum_{k=0}^{m}\binom{m}{k}\frac{x^k}{(1+x)^m}f\left(\frac{k}{m+1-k}\right)\quad (x\ge 0),$$

can be obtained from the operator $P_m^{(0,1)}$ (see [15], [1], [5] and [2]) using the rational transformation $t=\frac{x}{1+x}$ $(x\geq 0)$. By means of this transformation the operator (1.3) leads to a BBH type operator having the nodes: $x_{m,k}^{c,d}=(k+c)/$ $/(m+d-k-c)$, where $0\leq c\leq d$. If we choose $c=a$ and $d=a+1$ $(a\geq 0)$, then we obtain a BBH type operator investigated recently by O. Agratini [5]:

$$\left(L_m^{(\alpha)}f\right)(x)=\sum_{k=0}^{m}\binom{m}{k}\frac{x^k}{(1+x)^m}f\left(\frac{k+a}{m+1-k}\right).$$

**1.6.** Concerning the remainder of the approximation formula

$$f(x)=\left(P_m^{(c,d)}f\right)(x)+\left(R_m^{(c,d)}\right)(x), \tag{1.6}$$

in the paper [19] there was established the following representation, in terms of first and second order divided differences:

$$\left(R_m^{(c,d)}f\right)(x)=\frac{dx-c}{m+d}\sum_{k=0}^{m}p_{m,k}(x)\left[x,\frac{k+c}{m+d};f\right]-$$

$$-\frac{mx(1-x)}{(m+d)^2}\sum_{k=0}^{m-1}p_{m-1,k}(x)\left[x,\frac{k+c}{m+d},\frac{k+c+1}{m+d};f\right].$$

In the case $c=d=0$, when (1.3) becomes the Bernstein polynomial $B_m f$, it reduces to an expression obtained already in 1964 by the first author [18]:

$$(R_m f)(x)=\frac{x(x-1)}{m}\sum_{k=0}^{m}p_{m-1,k}(x)\left[x,\frac{k}{m},\frac{k+1}{m};f\right].$$

In a recent paper O. Agratini [4] studied the monotonicity properties of the sequence of polynomials (1.3).

In order to investigate the simultaneous approximation properties of the operator (1.3) there was established in [20] the following formula

$$\left(P_m^{(c,d)}f\right)^{(r)}(x)=m^{[r]}\sum_{j=0}^{m-r}p_{m-r,j}(x)\left(\Delta^r_{\frac{1}{m+d}}f\right)\left(\frac{j+c}{m+d}\right),$$

where $0\leq r\leq m$. By using it, there was proved the following result: *if $f\in C^r[0,1]$ then we have*

$$\lim_{m\to\infty}\left(P_m^{(c,d)}f\right)^{(r)}=f^{(r)},$$

*uniformly on the interval* $[0,1]$.



## 2. THE MEAN SQUARE ERROR OF THE OPERATOR $P_m^{(c,d)}$

**2.1.** Since the rate of convergence of the operators (1.3) is characterized by the value of the mean square error

$$e_m^2(x;c,d)=P_m^{(c,d)}\left((t-x)^2;x\right), \tag{2.1}$$

we next make an examination of it.

It is obvious that we can write

$$e_m^2(x;c,d)=P_m^{(c,d)}\left(t^2;x\right)-2xP_m^{(c,d)}(t;x)+x^2,$$

According to (1.5) we have

$$e_m^2(x;c,d)=\frac{mx(1-x)+(dx-c)^2}{(m+d)^2}. \tag{2.2}$$

The variance of the operator $P_m^{(c,d)}$ is defined by

$$v_m(x;c,d)=P_m^{(c,d)}\left(e_1-P_m^{(c,d)}e_1\right)^2(x)=\left(P_m^{(c,d)}e_2\right)(x)-\left[\left(P_m^{(c,d)}e_1\right)^2(x)\right]^2.$$

If we take into consideration the identities (1.5) we obtain

$$v_m(x;c,d)=\frac{mx(1-x)}{(m+d)^2}$$

and one observes that it does not depend on the parameter $c$.

**2.2.** In order to see how well a function $f\in C[0,1]$ can be approximated by the polynomial (1.3), we need to find the maximum value of (2.2) on the interval $[0,1]$.

We shall now present a basic result of this paper.

THEOREM 2.1. *If $m>d^2$, then the maximum value on $[0,1]$ of the mean square error (2.2) can be represented under the form*

$$M_m^{(c,d)}=\frac{m}{4(m+d)^2}\left[1+\frac{(d-2c)^2}{m-d^2}\right]. \tag{2.3}$$

*Proof.* It is known that if we have a polynomial of second degree, with real coefficients: $P_2(x)=Ax^2+Bx+C$ and $A<0$, then the maximum value of this polynomial is given by $P_2\left(-\frac{B}{2A}\right)=-\frac{\Delta}{4A}$, where $\Delta=B^2-4AC$. In our case we have $A=d^2-m$, $B=m-2cd$, $C=c^2$. Consequently, we find that this maximum value is given by

$$M_m^{(c,d)}=\frac{m^2-4c(d-c)m}{4(m+d)^2(m-d^2)}=\frac{m}{4(m+d)^2}\left[1+\frac{(d-2c)^2}{m-d^2}\right].$$

Now it is clear that we can formulate an important consequence of this theorem.

COROLLARY 2.1. *The least maximum value (2.2) is attained for $d=2c$ ($c\geq 0$) and it is*

$$\nu_m(c)=M_m^{(c,2c)}=\frac{m}{4(m+2c)^2}\leq\frac{1}{4m}=\nu_m(0).$$

In this case we have the approximating polynomials

$$\left(P_m^{(c,2c)}f\right)(x)=\left(S_m^{(c)}f\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)f\left(\frac{k+c}{m+2c}\right)\quad(c\geq 0),\tag{2.4}$$

which are important in numerical integration of functions.

## 3. A BEST QUADRATURE FORMULA

**3.1.** By using the approximation formula (1.6) we can obtain the following numerical quadrature procedure

$$\int_0^1 f(x)\mathrm{d}x=\frac{1}{m+1}\sum_{k=0}^{m}f\left(\frac{k+c}{m+d}\right)+r_m^{(c,d)}(f),\tag{3.1}$$

because

$$\int_0^1 p_{m,k}(x)\mathrm{d}x=\binom{m}{k}\frac{\Gamma(k+1)\Gamma(m-k+1)}{\Gamma(m+2)}=\frac{1}{m+1}.$$

For the monomials $e_0$, $e_1$, $e_2$ the remainder of (3.1) takes the values

$$r_m^{(c,d)}(e_0)=0,\quad r_m^{(c,d)}(e_1)=\frac{d-2c}{2(m+d)},$$

$$r_m^{(c,d)}(e_2)=\frac{m+6c\,(m+c)-2d(2m+d)}{6(m+d)^2}.$$

It is easy to see that if we choose $d=2c$ then the degree of exactness of the corresponding quadrature formula is $N=1$, although the operator $P_m^{(c,2c)}$ does not reproduce the linear functions.

This can be written under the following form



$$\int_0^1 f(x)\mathrm{d}x=\frac{1}{m+1}\sum_{k=0}^{m}f\left(\frac{k+c}{m+2c}\right)+K_m(c)f''(\xi),\tag{3.2}$$

where $0<\xi<1$ and

$$K_m(c)=-\frac{m-2c\,(m+c)}{12(m+2c)^2}=\frac{(2c-1)m+2c^2}{12(m+2c)^2}.$$

In the special case $c=0$ formula (3.2) will be

$$\int_0^1 f(x)\mathrm{d}x=\frac{1}{m+1}\sum_{k=0}^{m}f\left(\frac{k}{m}\right)-\frac{1}{12m}f''(\xi)\tag{3.3}$$

and it corresponds to the approximation of the function $f$ by the Bernstein polynomial $B_mf$.

**3.2.** One observes that the least value of $K_m(c)$ is obtained for $c=\frac{1}{2}$, when formula (3.2) becomes

$$\int_0^1 f(x)\mathrm{d}x=\frac{1}{m+1}\sum_{k=0}^{m}f\left(\frac{2k+1}{2m+2}\right)+\frac{1}{24(m+1)^2}f''(\xi)\tag{3.4}$$

which represent the well known composite midpoint or rectangular quadrature formula.

Therefore by using the Bernstein type polynomial

$$\left(P^{\left(\frac{1}{2},1\right)}f\right)(x)=(S_mf)(x)=\sum_{k=0}^{m}p_{m,k}(x)f\left(\frac{2k+1}{2m+2}\right)\tag{3.5}$$

we obtain the best quadrature formula having the form (3.1), namely formula (3.4).

In this case the least maximum value of the mean square error is $\nu_m\left(\frac{1}{2}\right)=\frac{m}{4(m+1)^2}$.

## 4. QUANTITATIVE ESTIMATIONS OF APPROXIMATION

**4.1.** Now we establish some estimates of the order of approximation of a function $f\in C[0,1]$ by means of the polynomials (1.3).

Since the constants are reproduced by the operator defined at (1.3), according to a known result (see, e.g., [17] or [12]), we can write

$$\left|f(x)-\left(P_m^{(c,d)}f\right)(x)\right|\le\left[1+\gamma^{-2}P_m^{(c,d)}\left((t-x)^2;x\right)\right]\omega_1(f;\gamma),$$

where $\omega_1$ represents the first order modulus of continuity and $\gamma>0$.

If we take into account (2.1) and (2.2), we obtain

(4.1) $$\left|f(x)-\left(P_m^{(c,d)}f\right)(x)\right|\le\left[1+\gamma^{-2}e_m^2(x;c,d)\right]\omega_1(f;\gamma).$$

Now we can state the following important result.

THEOREM 4.1. *If $m>d^2$, then in the sup norm we can write the inequality*

(4.2) $$\left\|f-P_m^{(c,d)}f\right\|\le\left\{1+\frac{m}{4(m+d)}\left[1+\frac{(d-2c)^2}{m-d^2}\right]\right\}\omega_1\left(f;\frac{1}{\sqrt{m+d}}\right).$$

*Proof.* The idea here is to take into consideration the fact that the maximum value of (2.2) is given at (2.3) and then to choose $\gamma=1/\sqrt{m+d}$.

In the special case of the operator $S_m^{(c)}$, defined at (2.4), we obtain the inequality

$$\left\|f-S_m^{(c)}\right\|\le\left[1+\frac{m}{4(m+2c)}\right]\omega_1\left(f;\frac{1}{\sqrt{m+2c}}\right).$$

It can be written also under the following form

(4.3) $$\left\|f-S_m^{(c)}f\right\|\le\left(\frac{5}{4}-\frac{c}{2(m+2c)}\right)\omega_1\left(f;\frac{1}{\sqrt{m+2c}}\right).$$

For $c=0$ it reduces to the inequality of Popoviciu-Lorentz ([16], [14]):

(4.4) $$\left\|f-B_mf\right\|\le\frac{5}{4}\omega_1\left(f;\frac{1}{\sqrt{m}}\right).$$

In the case of the polynomial $S_mf$, defined at (3.5), we find the following inequality

$$\left\|f-S_mf\right\|\le C_m\omega_1\left(f;\frac{1}{\sqrt{m+1}}\right),$$

where $C_m=\frac{1}{4}\left(5-\frac{1}{m+1}\right)$.

If we replace in (4.1) $\gamma=\alpha\,\delta$ $(\alpha,\delta\in\mathbb{R}_+)$, we obtain

$$\left|f(x)-\left(P_m^{(c,d)}f\right)(x)\right|\le\left[1+(\alpha\delta)^{-2}e_m^2(x;c,d)\right]\omega_1(f;\alpha\delta).$$



In the case $c=d=0$, by selecting $\delta=\sqrt{\dfrac{x(1-x)}{m}}$, we get

$$\left|f(x)-(B_mf)(x)\right|\le\left(1+\frac{1}{\alpha^2}\right)\omega_1\left(f;\alpha\sqrt{\frac{x(1-x)}{m}}\right).$$

This inequality permits to see that, indeed, the Bernstein polynomials are interpolatory in $x=0$ and $x=1$.

Because on [0, 1] we have $x(1-x)\le\frac{1}{4}$, by choosing $\alpha=2$, we arrive at the classical inequality (4.4).

**4.2.** Now let us use the second order modulus of smoothness

$$\omega_2(f;\gamma)=\sup\{|f(x-h)-2f(x)+f(x+h)|:x,x\pm h\in[a,b],0\le h\le\gamma\},$$

where $0\le\gamma\le\frac{1}{2}(b-a)$.

By using both moduli $\omega_1$ and $\omega_2$ one can find estimates of the approximation of the function $f$ by means of the operator (1.3).

For this purpose we can use an inequality of H. H. Gonska and R. K. Kovacheva [13], included in

LEMMA 4.1. *If $K=[a,b]$ is a compact interval of the real axis and $K'=[a',b']$ is a subinterval of it, and if we assume that $L:C(K)\to B(K')$ is a positive operator, such that $L(1;x)=1$ and $0<\gamma<\frac{1}{2}(b-a)$, then we have*

$$\left|f(x)-L(f(t);x)\right|\le\frac{2}{\gamma}\left|L(t-x;x)\right|\omega_1(f;\gamma)+$$
$$+\left[\frac{3}{2}+\frac{3}{2\gamma}\left|L(t-x;x)\right|+\frac{3}{4\gamma^2}L\left((t-x)^2;x\right)\right]\omega_2(f;\gamma).$$

If we take into account the relations (1.5) and (2.2) we obtain the inequality

(4.5) $$\left|f(x)-\left(P_m^{(c,d)}f\right)(x)\right|\le\frac{2}{\gamma}\cdot\frac{|c-dx|}{m+d}\omega_1(f;\gamma)+$$
$$+\left[\frac{3}{2}+\frac{3}{2\gamma}\cdot\frac{|c-dx|}{m+d}+\frac{3}{4\gamma^2}e_m^2(x;c,d)\right]\omega_2(f;\gamma).$$

This implies the following

$$\left\|f-P_m^{(c,d)}\right\|\le\frac{2}{\gamma}\cdot\frac{d}{m+d}\omega_1(f;\gamma)+$$

$$+\left\{\frac{3}{2}+\frac{3d}{2\delta(m+d)}+\frac{3m}{4(m+d)^2\delta^2}\left[1+\frac{(d-2c)^2}{m-d^2}\right]\right\}\omega_2(f;\gamma).$$

If we choose $\gamma=1/\sqrt{m+d}$, then we get

$$\left\|f-P_m^{(c,d)}\right\|\le\frac{2d}{\sqrt{m+d}}\,\omega_1\left(f;\frac{1}{\sqrt{m+d}}\right)+$$

$$+\left\{\frac{3}{2}+\frac{3d}{2\sqrt{m+d}}+\frac{3m}{4(m+d)}\left[1+\frac{(d-2c)^2}{m-d^2}\right]\right\}\omega_2\left(f;\frac{1}{\sqrt{m+d}}\right).$$

For the operator $S_m^{(c)}=P_m^{(c,2c)}$ we obtain

$$\left\|f-S_m^{(c)}f\right\|\le\frac{4c}{\sqrt{m+2c}}\,\omega_1\left(f;\frac{1}{\sqrt{m+2c}}\right)+$$

$$+3\left[\frac{1}{2}+\frac{c}{\sqrt{m+2c}}+\frac{m}{4(m+2c)}\right]\omega_2\left(f;\frac{1}{\sqrt{m+2c}}\right).$$

In the case $c=0$ we get the estimation: $\|f-B_mf\|\le\frac{9}{4}\omega_2\left(f;\frac{1}{\sqrt{m}}\right)$.

But if we replace in (4.5) $\gamma=\alpha\delta$ $(\alpha,\delta\in\mathbb{R}_+)$, in the case $c=d=0$, we find

$$|f(x)-(B_mf)(x)|\le\frac{3}{4}\left(2+\frac{1}{\alpha^2}\right)\omega_2\left(f;\alpha\sqrt{\frac{x(1-x)}{m}}\right).$$

Selecting the parameter $\alpha$ such that we have $\alpha\max\sqrt{x(1-x)}=1$ on $[0,1]$, we find that we have to take $\alpha=2$ and we arrive at the important inequality

$$\|f-B_mf\|\le C\omega_2\left(f;\frac{1}{\sqrt{m}}\right),$$

where $C=\frac{27}{16}=1.6875$.

This inequality was first given in 1994 in the work [13].

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