REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

Tome 28, N° 1, 1999, pp. 97–104

ON THE APPROXIMATE SOLUTION OF THE EXCLUSIVE CASE OF SINGULAR INTEGRAL EQUATIONS WITH COMPLEX CONJUGATION VALUES ON THE UNKNOWN FUNCTION ON THE CLOSED LIAPUNOV CONTOUR

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1. The formulation of the problem. Let Γ be a closed Liapunov curve, that limits the simple connected D^+ on the complex plane. Let us consider that the point $z = 0$ belongs to D^+.

We examine the singular integral equation (SIE) with conjugation on the contour Γ

$$(R\varphi \equiv) \, c_1(t)\,\varphi(t) + d_1(t)\frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\mathrm{d}\tau + c_2(t)\overline{\varphi}(t) + d_2(t)\frac{1}{\pi i}\overline{\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\mathrm{d}\tau} +$$

$$+\frac{1}{2\pi i}\int_\Gamma h_1(t,\tau)\,\varphi(\tau)\,\mathrm{d}\tau + \overline{\frac{1}{2\pi i}\int_\Gamma h_1(t,\tau)\,\varphi(\tau)\,\mathrm{d}\tau} = f(t), \quad t \in \Gamma, \tag{1}$$

where $c_k(t)$, $d_k(t)$, $h_k(t,\tau)$, $k = \overline{1,2}$ and $f(t)$ are given function on Γ and $\varphi(t)$ is unknown function.

Let us suppose, that equation (1) is one of non elliptic type, id est the function

$$\Delta(t) \overset{def}{=} [c_1(t) + d_1(t)][\overline{c}_1(t) - \overline{d}_1(t)] - [c_2(t) + d_2(t)][\overline{c}_2(t) - \overline{d}_2(t)]$$

has zeroes of integer order on Γ.

The last condition is equivalent to that

det $[C(t) \pm D(t)]$ has zeroes of integer order on Γ, where

$$C(t) = \begin{bmatrix} c_1(t) & c_2(t) \\ \overline{c}_2(t) & \overline{c}_1(t) \end{bmatrix}, \qquad D(t) = \begin{bmatrix} d_1(t) & -d_2(t) \\ \overline{d}_2(t) & -\overline{d}_1(t) \end{bmatrix}.$$

The publications [1–5] are dedicated to the approximate solution of equation (1) either in elliptic and non elliptic cases. In all these works the

1991 AMS Subject Clasification: 45G05.

theoretical foundation of approximate methods was obtained for the case, when equations are given on the unit circle Γ_0 of complex plane. But the case of more general contours such as Liapunov contour, wasn't studied in the scientific literature. This work has the purpose to cover this lack. There are elaborated and theoretically founded collocation and quadrature methods for SIE of nonelliptic type, given on an arbitrary Liapunov contour Γ and examined in Hölder functions spaces.

2. The computing schemes of the methods. Let us introduce new unknown functions $\varphi_1(t)=\varphi(t)$ and $\varphi_2(t)=\overline{\varphi}(t)$. Then, from the representation

$$\overline{S\varphi(t)} = -(S\overline{\varphi})(t)+(T_1\overline{\varphi})(t),$$

and

$$\overline{\frac{1}{2\pi i}\int_\Gamma h(t,\tau)\,\varphi(\tau)\,d\tau} = -\frac{1}{2\pi i}\int_\Gamma \overline{h}(t,\tau)\,\overline{\varphi}(\tau)\,\overline{(\tau'(\sigma))^2}\,d\tau,$$

where S is the singular operator on Γ, and T_1 is the singular integral operator with the kernel $\dfrac{\partial\theta(\sigma,s)}{\partial\sigma}$, $\theta=\arg(\tau-t)$, $\tau=\tau(\sigma)$ ($\sigma\in[0,l]$, l is length of Γ) is the contour Γ equation, σ is the arch's absciss, the equation (1) is reduced to the equivalent SIE system without conjugation

$$(M\Phi\equiv)\, C(t)\,\Phi(t)+D(t)\frac{1}{\pi i}\int_\Gamma \frac{\Phi(\tau)}{\tau-t}d\tau + \tag{2}$$
$$+\frac{1}{2\pi i}\int_\Gamma H(t,\tau)\,\Phi(\tau)\,d\tau = F(t),\quad t\in\Gamma,$$

where $C(t)$ and $D(t)$ are determined above, $H(t,\tau)$ is the matrix-function of second order, $F(t)$ and $\Phi(t)$ are the vector-functions of the second order:

$$H(t,\tau)=\begin{bmatrix} h_1(t,\tau) & -\overline{h_2}(t,\tau)\overline{(\tau')^2}+2d_2(t)\dfrac{\partial\theta}{\partial\sigma} \\ h_2(t,\tau) & -\overline{h_1}(t,\tau)\overline{(\tau')^2}+2d_1(t)\dfrac{\partial\theta}{\partial\sigma}\end{bmatrix},$$

$$\Phi(t)=\{\varphi_1(t),\varphi_2(t)\},\quad F(t)=\{f(t),\overline{f(t)}\}.$$

In the system (2) the vector-function $\Phi(t)=\{\varphi_1(t),\varphi_2(t)\}$ is unknown. According to [1, p. 38], the function $\eta(\sigma,s)=\dfrac{\partial\theta(\sigma,s)}{\partial\sigma}$ is given in the form $\eta(\sigma,s)=\dfrac{K(\sigma,s)}{|\sigma-s|^\lambda}$, $0\le\sigma,s\le l$, $\lambda\in(1-\mu;1)$, μ is the exponent of contour's Γ

smoothness, and $K(\sigma, s)$ satisfies the Hölder condition for both variables with exponent μ.

The equation (1) and the system (2) are equivalent in the sense (see [1]) that for every solution $\varphi(t)$ of equation (1) there is a corresponding solution $\Phi=\{\varphi(t),\overline{\varphi}(t)\}$ of the system (2) expressed by the function $\varphi(t)$, and conversely, if $\Phi(t)=\{\varphi_1(t),\varphi_2(t)\}$ is the solution of system (2), then the corresponding solution of equation (1) can be found by the formula:

$$\varphi(t)=\frac{1}{2}\left[\varphi_1(t)+\overline{\varphi}_2(t)\right]. \tag{3}$$

Further we consider, that functions $c_k(t)$ and $d_k(t)$ ($k=1,2$) belong to the space $H_\alpha^{(q+1)}(\Gamma)$, $h_k(t,\tau)\in H_\alpha^{(q)}(\Gamma)$ by both variables, $f(t)\in H_\alpha^{(q+1)}(\Gamma)$, where q is determined below. Moreover, we consider the following representation to be true:

$$\begin{aligned} C(t)+D(t)&=A(t)=A_1(t)D_-(t)R_-(t),\\ C(t)-D(t)&=B(t)=B_1(t)D_+(t)R_+(t),\end{aligned} \tag{4}$$

where $\det A_1(t)\det B_1(t)\neq 0$, $t\in\Gamma$, $R_\pm(t)$ are polynomial matrix-functions by $t^{\pm 1}$ with constant and nonzero determinants, and the matrix-functions $D_+(t)$ and $D_-(t)$ have the following form:

$$D_+(t)=\left\{\prod_{k=1}^{s}(t-\beta_k)^{\nu_j^{(k)}}\delta_{jk}\right\}_{j,k=1}^{2},$$

$$D_-(t)=\left\{\prod_{k=1}^{p}\left(\frac{1}{t}-\frac{1}{\alpha_k}\right)^{\mu_j^{(k)}}\delta_{jk}\right\}_{j,k=1}^{2},$$

where δ_{jk} is the Kroneker symbol, α_k, $k=\overline{1,p}$, β_k, $k=\overline{1,s}$ are contour's Γ points, $\mu_1^{(k)}\ge\mu_2^{(k)}\ge 0$, $k=\overline{1,p}$, $\nu_1^{(k)}\ge\nu_2^{(k)}\ge 0$, $k=\overline{1,s}$ are natural numbers.

Let us denote

$$q=\max\left(\mu_1^{(1)},\mu_2^{(2)},\nu_1^{(1)},\nu_2^{(2)}\right). \tag{5}$$

For such functions, as $c_k(t)$ and $d_k(t)$, $k=\overline{1,2}$ the equation (1) is degenerated. Further we consider that left particular indexes of matrix-function $B_1^{-1}(t)\,A_1(t)$ are equal to zero and the equation (1) has the unique solution.

The approximate solution of the equation (1) are found using the approximate solutions of the system (2). The approximate solution of the system (2) is seeking in the form

(6) $$\Phi_k(t)=\sum_{k=-n}^{n} c_k t^k, \quad t\in\Gamma,$$

where $c_k=\{c_k^{(1)},c_k^{(2)}\}$ are unknown vectors of dimension 2. Then, according to (3), the approximate solution of the equation (1) has the following form

(7) $$\varphi_n(t)=\frac{1}{2}\sum_{k=-n}^{n}\left(c_k^{(1)}t^k+\overline{c}_k^{(2)}\overline{t}^k\right), \quad t\in\Gamma.$$

a) *Collocation method.* The approximate solution of the system SIE (2) is seeking in the form (6), and the unknowns c_k, $k=\overline{-n,n}$ are determined from condition of turning into zero of the error $(M\Phi)(t)-f(t)$ in points t_j $(\in\Gamma)$, $j=\overline{0,2n}$:

$$(M\Phi)(t_j)-f(t_j)=0, \quad j=\overline{0,2n}.$$

It is easy to check, that the last conditions turn to the following system of linear algebraic equations (SLAE):

(8) $$\sum_{k=-n}^{n}\Big\{\left[A(t_j)\operatorname{sign}(k)+B(t_j)\operatorname{sign}(-k)\right]t_j^k+ +\frac{1}{2\pi i}\int_\Gamma H(t_j,\tau)\tau^k d\tau\Big\}c_k=F(t_j), \quad j=\overline{0,2n};$$

here $\operatorname{sign}(k)=1$, for $k\ge 0$ and $\operatorname{sign}(k)=0$, $k<0$.

b) *Quadrature method.* Taking into consideration that matrix-function $H(t,\tau)$ has an integrational singularity for $t=\tau$, the quadrature method can not be applied directly to the equation (2). Let us introduce a new system of SIE, closed to the equation (2), with the regular kernel without singularities. So, the quadrature method can be applied to the new equation. Let ρ be an arbitrary positive number. We'll denote by $\eta_\rho(\sigma,s)$ the function

(9) $$\eta_\rho(\sigma,s)=\begin{cases}\eta(\sigma,s), & \text{when } |\sigma-s|\ge\rho\\ \rho^{-\lambda}K(\sigma,s); & \text{when } |\sigma-s|<\rho,\end{cases}$$

and by $H_\rho(t,\tau)$ the matrix-function that is obtained from matrix-function $H(t,\tau)$ substituting the functions $\frac{\partial\theta}{\partial\sigma}$ by function $\eta_\rho(\sigma,s)$.

Now let us examine the system of SIE

(10) $$(M_\rho x)(t)\equiv C(t)\,x(t)+\frac{D(t)}{\pi i}\int_\Gamma\frac{x(\tau)}{\tau-t}d\tau+ +\frac{1}{2\pi i}\int_\Gamma H_\rho(t,\tau)\,x(t)\,d\tau=F(t), \quad t\in\Gamma,$$

and apply the quadrature method. According to this method, the approximate solution of system of SIE (10) is seeking in the form (6), and the unknowns c_k, $k=\overline{-n,n}$ are obtained from SLAE (8), where function $H(t,\tau)$ is substituted by $H_\rho(t,\tau)$, and integrals $\frac{1}{2\pi i}\int_\Gamma H_\rho(t_j,\tau)\,\tau^k\,d\tau$, $j=\overline{0,2n}$; $k=\overline{-n,n}$ are substituted by some quadrature formula. As a quadrature formula, we will chose the following interpolation quadrature formula

(11) $$\frac{1}{2\pi i}\int_\Gamma\tau^k g(\tau)\,d\tau\cong\frac{1}{2\pi i}\int_\Gamma U_n[g(\tau)\tau]\tau^{k-1}\,d\tau, \quad k=\overline{-n,n}.$$

Here U_n is the interpolation operator by points t_j, $j=\overline{0,2n}$:

(12) $$(U_n g)(\tau)=\sum_{r=0}^{2n}g(t_r)l_r(\tau);$$
$$l_r(t)=\left(\frac{t_r}{t}\right)^n\cdot\prod_{k=0,k\ne r}^{2n}\frac{t-t_k}{t_r-t_k}=\sum_{s=-n}^{n}\Lambda_s^{(r)}t^s, \quad r=\overline{0,2n}.$$

Taking into consideration the formulas

$$\frac{1}{2\pi i}\int_\Gamma\tau^r\,d\tau=\begin{cases}0, & \text{when } r\ne -1,\\ 1, & \text{when } r=-1,\end{cases}$$

and the definition (12), it is easy to show that the quadrature formula (11) turns into following:

(13) $$\frac{1}{2\pi i}\int_\Gamma\tau^k g(\tau)\,d\tau\cong\sum_{r=0}^{2n}g(t_r)\,t_k\,\Lambda_{-k}^{(r)}, \quad k=\overline{-n,n}.$$

So, the SLAE for quadrature method is the following:

(14) $$\sum_{k=-n}^{n}\Big[C(t_j)t_j^k+\operatorname{sign}(k)D(t_j)t_j^k+ \sum_{r=0}^{2n}H_\rho(t_j,t_r)\,t_r\,\Lambda_{-k}^{(r)}\Big]\alpha_k=f(t_j), \quad j=\overline{0,2n}.$$

3. The formulation of the basic theorems. The theorems that give the theoretical foundation of computing schemes elaborated in p. 2 are given below.

THEOREM 1. *Let $c_k(t)$ and $d_k(t)$, $k = 1, 2$ belong to space $H_\alpha^{(q+1)}(\Gamma)$, $0<\alpha\le 1$, $h_k(t,\tau)$, $k = 1, 2$, $f(t)\in H_\alpha^{(q)}(\Gamma)$, the number q is determined by connection (5) and representations (4) are true. If $\det A_1(t)\ne 0$, $\det B_1(t)\ne 0$, $t\in\Gamma$, the left particular indexes of the matrix-functions $B_1^{-1}(t)\,A_1(t)$ are equal to zero, the equation (1) has a unique solution and points t_j, $j=\overline{0,2n}$ form a system of Fejer nodes on Γ, then for sufficiently large values of numbers n, SLAE (8) of collocation method can be solved in the unique order and the approximate solutions $\varphi_n(t)$ from (7), converge in the space $H_\beta(\Gamma)$, $0<\beta<\delta$ to the exact solution $\varphi(t)$ of the equation (1) with the rate*

(15) $$\|\varphi-\varphi_n\|_\beta = O\left(\frac{\ln n}{n^{\delta-\beta}}\right),$$

where $\delta=\min(\alpha,\mu)$, μ is the exponent of the contour's Γ smoothness.

THEOREM 2. *Let all conditions of theorem 1 be fulfilled and, moreover, $\beta\in(0;\nu)$, $\nu=\min(\alpha;\mu;1-\mu)$. Then, for sufficiently large values of n and sufficiently small values of ρ, the SLAE (14) of quadrature method has a unique solution $c_k^{(\rho)}=\{c_{k1}^{(\rho)};c_{k2}^{(\rho)}\}$.*

The approximate solutions $\varphi_{n,\rho}(t)$ obtained by the formula

$$\varphi_{n,\rho}=\frac{1}{2}\sum_{k=-n}^{n}\left(c_{k1}^{(\rho)}t^k+\overline{c}_{k2}^{(\rho)}\overline{t}^k\right),$$

converge in space $H_\beta(\Gamma)$ to the exact solution $\varphi(t)$ of the equation (1) so that

(16) $$\lim_{n\to\infty}\lim_{\rho\to 0}\|\varphi-\varphi_{n,\rho}\|_\beta=0.$$

The following estimations for the convergence rate are true:

(17) $$\|\varphi-\varphi_{n,\rho}\|_\beta=O\left(\frac{\ln^2 n}{n^{\nu-\beta}}\right)+O(\rho^\gamma),\quad \gamma=\min(\mu,1-\mu).$$

The proof of these theorems is made according to results of [2] (Chapter 11, p. 442–444) and [3, Chapter 2].

The proof of theorem 1. It easy to check, that SLAE (8) of collocation method is equivalent to operatorial equation

(18) $$(U_nMU_n\Phi_n\equiv)U_n(AP+BQ+K)U_n\Phi_n=U_nF,$$

examined as an equation in a limit-dimensional space $U_n[H_\beta(\Gamma)]_2$, concerning the unknown two-dimensional vector-function $\Phi_n(t)$; K is integrational operator with kernel $H(t,\tau)$.

Do to the representations (4), the equation (18) is equivalent to the operatorial equation (in the same space)

(19) $$U_n\left\{\left(PC_-+QC_+^{-1}\right)V+T_1V+K_1\right\}U_n\Phi_n=U_nF_1,$$

where matrix-functions $C_+(t)$, $C_-(t)$ are the coefficients of left canonical factorization of matrix-function $B_1^{-1}(t)\,A(t)$:

$$B_1^{-1}(t)\,A(t)=C_+(t)\,C_-(t),\quad t\in\Gamma,$$

$$V=PD_-R_-I+QD_+R_+I,\quad T=QC_-P+PC_+^{-1}Q,$$

$$K_1=C_+^{-1}B_1^{-1}K+C_-QD_-R_-P+C_+^{-1}PD_+R_+Q,\quad F_1=C_+^{-1}B_1^{-1}F,$$

where P and Q are Riesz projectors $P=\frac{1}{2}(I+S)$, $Q=I-P$.

Following [2, p. 442–443] let us introduce the space

$$X_V=\left\{z(t)\in\left[H_\beta(\Gamma)\right]_2;\ (Vz)\in\left[H_\beta^{(q)}(\Gamma)\right]_2(\equiv Z)\right\}.$$

The norm of this space is

$$|z|=|Vz|_Z=\sum_{i=0}^{q}\left\|(Vz)^{(i)}\right\|_C+\left\|(Vz)^{(q)}\right\|_\beta$$

This space become a Banach space, $Z\subset X_V$. It is easy to check, that $\ker V=0$. That's why the contraction $\tilde V=V/X_V$ is one-to-one continuous mapping X_V on Z. Let further $E:X_V\to[H_\beta(\Gamma)]_2$ be the embedding operator. Then, for the contraction $\tilde M$ of the operator M on the space X_V we obtain representation $\tilde M=M_0+T_0$, where $M_0=(PC_-+QC_+^{-1})\tilde V$, $T_0=(T+K_1)E$.

As the operator $PC_-+QC_+^{-1}:[H_\beta(\Gamma)]_2\to Z$ is continuously inversible, then the operator $M_0:X_V\to Z$ is also continuously inversible and the operator $T_0:X_V\to[H_\delta^{(q+1)}]_2$ is continuously inversible. Let denote by $\tilde U_n$ the contraction U_n on X_V.

Repeating now the reasoning from [3, §10], we obtain that for sufficiently large values of numbers $n\ge n_1$ the operator $U_nM_0\tilde U_n:\tilde U_nX_V\to U_nZ$ is inversible, and moreover:

$$\left\|(U_n M_0 \tilde{U}_n)^{-1}\right\| = O(1).$$

Then, due to the total continuity of $T_0 : X_V \to [H_\delta^{(q+1)}(\Gamma)]_2$, the operator $U_n(M_0 + T_0)\tilde{U}_n : \tilde{U}_n X_V \to U_n Z$ is also inversible for $n \geq n_2$ ($\geq n_1$), and moreover

(20) $$\left\|[U_n(M_0 + T_0)\tilde{U}_n]^{-1}\right\| = O(1).$$

Consequently, both operatorial equation (19) and equation (18) have unique solutions. This is the proof of the uniqueness of the solution of SLAE (8).

Let us determine the estimation of the convergence (15) to approximate solution of the exact solution of equation (1). In accordance with (20) and with estimation.

$$\|g - U_n g\|_\beta = O\left(\frac{\ln n}{n^{\delta-\beta}}\right),$$

that take place according to the chosen interpolation kernel [3, §6], it is easy to obtain that

$$\|\Phi_n - \Phi\|_\beta = O\left(\frac{\ln n}{n^{\delta-\beta}}\right).$$

Remains to use the formulas (3) and (7).

Theorem 1 is proved.

Theorem 2 is proved in accordance with results that are obtained while proving the theorem 1, and that are used for the general theory of direct, but not projection methods determined in [3, §1], and using the accuracy of quadrature formula (11) obtained in [3, §8].

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Received: December 8, 1998